



Hugh Mitton, CC BY 3.0 via Wikimedia Commons



Why build bridges?

... when you can burn it all down?

https://xkcd.com/386/





Why build bridges?



Mathematicians? Mathematics educators? ... Building bridges?

Main argument:





For the **functioning of mathematics education**

we need to seek out and embrace **tension in perspectives**



Making the argument Part 1 of 2

We seek out opposing perspectives because it is **who we are** and **who we need to be**



Building bridges shapes **questions we see** influences **our approaches**



We seek out opposing perspectives because it is **who we are** and **who we need to be**

3 examples

Example 1. Göttingen, late 1800s/early 1900s





Naming a problem in 1908 that we still work on today

In *Elementary Mathematics from an Advanced Standpoint*, Felix Klein described what he called the double discontinuity experienced by prospective high school teachers:

> The year g ariversity state it [wes] confronted with protients that did not suggest ... the things with which he had been concerned at school. When, after finishing his course of study, he became a teacher ... he was scarcely able to discern any <u>connec-</u>

> > tween his task and his university mathematics.¹

There's double discontinuity. We should do something about it. iolt experienced by the hi

The Mathematical Education of Teachers II

Volume 17

 BMS

1908

E Contra Mahemati Society Materiatical Association of Auerta 2012 The young university student, at the beginning of his studies, is confronted with problems which do not at all remind him of the things that he studied at school (as a result, of course, he forgets these things rapidly and thoroughly). If after completing his university studies he becomes a teacher, then he suddenly needs to teach this ordinary elementary mathematics in a manner which is appropriate for school; since on his own he can hardly relate this task to his university mathematics, in most cases he will quickly adopt the time-honored traditional pedagogy, and the university studies remain a more or less pleasant memory, which has no influence on his teaching.

School University mathematical mathematical practice practice **Structures**? **Intuitions? Branches of mathematics?** Approaches to concepts **Student conceptions** School University pedagogical pedagogical practice practice

with thanks to Michael Hutchings and Juan Souto for the translation

Klein's capacity to forge alliances among mathematicians, school teachers, and engineers showed grace and curiosity in the face of conflict.

13(2)



Example 2. US, 1990's to 2000s



Estimate 2^{π} . Show your estimate on the number line below.

Estimate 2^{π} . Show your estimate on the number line below.



What are 3 different right triangles whose hypotenuse has length $\sqrt{2}$?





In contrast, consider the following ...

 $2^{\pi} = 2^3 \ 2^{0.1} \ 2^{0.04} \ 2^{0.001} \dots$ These numbers get close to 0, so it is close to 0

Mathematical knowledge **for teaching** What is the student's reasoning? How would you respond?

0 1 2

(Ball & Bass, 2002; Shulman, 1987; Thompson & Thompson, 1996)

Mathematical knowledge **for teaching**

(Ball & Bass, 2002; Shulman, 1987; Thompson & Thompson, 1996)

What if the rounding leads to highly inaccurate estimates? What would you do? Why?

 $(1.26)^{2} + \chi^{2} = 2$ $\chi^{2} = 2 - 1.69$

 $\chi = 0.64$

sin 60° = s

y 1.26

12

0.64

Mathematical knowledge **for teaching**

What mathematics could this question lead to?

How are lottery prices determined?

What would you say or do help students say their ideas to each other, precisely yet in their own language?

What knowledge do you need to enact these moves?

The notion of

Mathematical Knowledge for Teaching

was first developed in the 1990s-2000s

During this period ...



If we continue down this road of infighting ... the real losers will be the students of America

– Secretary of Education Richard Riley, at the 1998 Joint Mathematics Meetings



What mathematical knowledge is entailed in recurrent work of teaching?

Deborah Ball, Hyman Bass, and colleagues

Mathematical Knowledge for Teaching

is one existence proof of:

- the power of seeking out opposing perspectives
- finding complementarity in mathematics and education research
- intellectual work possible only with both
- systemic reshaping of entire programs and policies possible only with both



Example 3. Worldwide, 2000s-now



Abstract algebra Mathematical modeling Modern geometry Real analysis Statistics

> Functions Euclidean geometry Number & operation Posing problems about the world around us

Contemporary work to mend discontinuity








MODULE(S²)

Mathematics Of Doing, Understanding, Learning, and Educating for Secondary Schools



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ALGEBRA GEOMETRY STATISTICS MATHEMATICAL MODELING

NODULE(S²) www.modules2.com

GoalTo enhance mathematics faculty capacity to develop
prospective teachers' MKT in content coursesActivitiesDeveloped materials for 4 semester-long coursesProfessional development for mathematics faculty
Data-driven improvement
Research on MKT at the secondary level

Universal Algebra **Statistics Education High School Teaching Operations Research** Infinitary Combinatorics Coxeter Groups Computational Fluid Dynamics Mathematical Modeling Education Mathematical Knowledge for Teaching Mathematics Teacher Learning

Modelling Biological Systems





























Contemporary work to mend double discontinuity is one existence proof of:

Results through seeing mathematics and mathematics education research

as complements

Why build bridges? We seek out opposing complementary perspectives because it is who we are and who we need to be

to shape the **questions we see** to influence **our approaches**

Part 2

The alternative to seeking out complementary perspectives is math wars

So what?

Joint work with Erin E. Baldinger, Katherine Burke, Charles E. Wilkes II

What is the cost of math wars?

Math wars lead to **loss of trust** and **lost intellectual work**.

Inversely, if we can learn through tensions in perspectives, we have the potential to make progress that we could not have individually made.

Math Wars?

America's maths wars INOV 6th 2021 WASHINGTON, DC THE NEW YORKER HOW MATH BECAME AN **OBJECT OF THE CULTURE WAR***S*

The Economist

United States | Problem-solving

As was true in the nineties, today's fights about math are not entirely about what kids actually learn in their classrooms.

Math Wars?







Math Wars: Context & Proposed Definition







1989 NCTM "Standards" **1991** California Math Framework **1997** California Math Standards

Math Wars: Context & Proposed Definition



Math Wars: Context & Proposed Definition

The presence of public discourse that
(1) addresses mathematics education
(2) dismisses opposing points of view using extreme, binary, and/or inflammatory language, and
(3) occurs over a sustained duration of time.

What has been assumed:

"therefore"

Math Wars are divisive; therefore, children suffer.

What we ask:

What is the cost of the act of Math Wars?

mathematics education educational psychology number theory topology markov processes category theory PDEs

Argument

Math wars lead to loss of trust in the enterprise of mathematics education, and to constricted boundary spaces.

> Inversely, if we can learn through tensions in perspectives, we have the potential to make progress that we could not have individually made.

Institutional trust

A professional individual's belief that another professional individual/group will generally act in ways consistent with the individual's expectations of positive behavior

OECD, 2017

Or a mathematician just said, "I wanted to help, but they all looked at me like I didn't have the right to say anything because I wasn't a teacher." Or mathematicians feeling like they were cast as part of the evil empire. Or teachers feeling like they were cast as not knowing enough to be able to say anything about what mattered or not mattering enough.

– Elaine (Teacher Education Researcher)

★ all names are pseudonyms

And you don't really know who's right or who's wrong. [...] And suddenly we just don't feel a sense of security about what you're learning [...] Or who to trust.

[It's like] the grown ups have left the room.

– Ruth (Mathematician)

Institutional trust

A professional individual's belief that another professional individual/group will generally act in ways consistent with the individual's expectations of positive behavior

OECD, 2017

Casual hate and disregard

- "evil" / "devil" / "automatically the bad guy"
 - Elaine, Kay, Victor
- "nasty" / "nastiness" Sam, Elaine, Nell

Public fighting

• "instability" – Ruth, Eric

Boundary space

Norms, processes, and interactions that support learning and creation across discontinuities between communities

Akkerman & Bakker, 2011

If people hadn't been alienated, and therefore turned away from that kind of work, more progress could have been made. I think there was a lot of both lost energy, but lost opportunity. [...]

I think more interesting things would have happened. – Elaine (Teacher Education Researcher)

But if it devolves into – this person is not the right kind of person – it devolves into *ad hominem* arguments [...] it creates an uninviting atmosphere. – Sam (Mathematician/Education Researcher)

Mathematicians didn't acknowledge the epistemology of educational research and the educational researchers quit acknowledging the importance of the mathematical perspective on that.

– Nell (Education Researcher)

Boundary space

Norms, processes, and interactions that support learning and creation across discontinuities between communities Akkerman & Bakker, 2011

Heightened risk to identity

- "uninviting"
- "alienated"

Boundary work is impossible without professional identity and identities

Argument

Math wars lead to loss of trust in the enterprise of mathematics education, and to constricted boundary spaces.

> Inversely, if we can learn through tensions in perspectives, we have the potential to make progress that we could not have individually made.

The next exposure I had to math wars was really in the late '80s, early '90s, when I was asked to be a reviewer for the Systemic Initiative grant. ... We did use to have ... some knock down drag out for a while, but we're friends. I think we really did have a tremendous amount of respect for each other. Right?

– Ruth (Mathematician)

And just you know it's like you'd say this one word but it came with this huge package of things. [---] being the person that she is would say, "I don't understand what that means." And all of a sudden we'd have to sit back and think, "Oh, what does it mean?"... once we all think that everybody knows what we're talking about, then it sort of begins to fall apart. ... the way I was involved in it I could see more positivity from it than negativity – Faith (Mathematics educator)

He just was not afraid to interact with anyone. And what I learned was, that's the only place you are really required to support your beliefs. Because if you're always with the people who agree with you, then you don't have to delve very deeply into what you're thinking. – *Faith*, about Jack

I don't know if I had an accurate sense of how awesome the people were that I wound up working with [...] All I can say is, man, was I lucky. – *Sam* (mathematician)

I can say very candidly that some people were not happy with me. – *Jack* (mathematics educator)

Even in math wars, there were glimmers of hope

It required taking risks to professional identity, and openness to seemingly opposing perspectives

so that mathematicians/educators could risk learning from one another

Part 3

What we face, now

Pop quiz: What do these have in common?

Relevance vs. Rigor Applied vs. Theoretical Concepts vs. Procedures Data Science vs. Algebra

Pop quiz: What do these have in common?



"Relevance" vs. "Rigor" "Applied" vs. "Theoretical" "Concepts" ". "Procedures" "Data Science" vs Algebra" "WE" . "OTHER"

We need to seek out opposing complementary perspectives

The alternative is **math wars**

whose cost is **loss of trust** and **loss progress** For the **functioning of our discipline**

we need to seek out and embrace tension in perspectives
Parting thoughts

Recast this narrative: "Mathematics = Gatekeeping"

We need to be open to the possibility of change, and that as a community, we can aim for equity. "Mathematician" "Mathematics" "Educator" "Education"

When will we use these terms not to divide and lament, but to honor, include, and learn?



Thank you!

Yvonne Lai

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