IBL in graduate math

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Online Seminar On Undergraduate Mathematics Education
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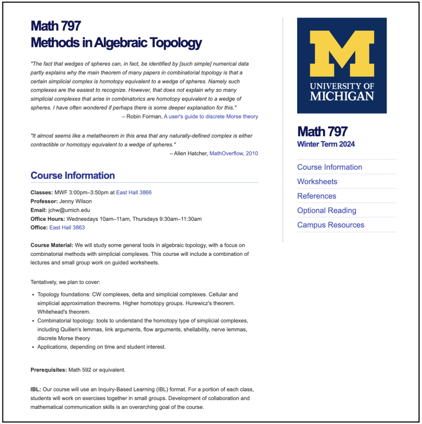
An experiment: An IBL graduate topics course

Math 797:
A class on methods in combinatorial topology

All course material available:

`https://dept.math.lsa.umich.edu/~jchw/2024Math797.html`

Class homepage:



Math 797
Methods in Algebraic Topology

"The fact that wedges of spheres can, in fact, be identified by [such simple] numerical data partly explains why the main theorem of many papers in combinatorial topology is that a certain simplicial complex is homotopy equivalent to a wedge of spheres. Namely such complexes are the easiest to recognize. However, that does not explain why so many simplicial complexes that arise in combinatorics are homotopy equivalent to a wedge of spheres. I have often wondered if perhaps there is some deeper explanation for this."

– Robin Forman, A user's guide to discrete Morse theory

"It almost seems like a metatheorem in this area that any naturally-defined complex is either contractible or homotopy equivalent to a wedge of spheres."

– Allen Hatcher, MathOverflow, 2010

Course Information

Classes: MWF 3:00pm–3:50pm at East Hall 3866
Professor: Jenny Wilson
Email: jchw@umich.edu
Office Hours: Wednesdays 10am–11am, Thursdays 9:30am–11:30am
Office: East Hall 3863

Course Material: We will study some general tools in algebraic topology, with a focus on combinatorial methods with simplicial complexes. This course will include a combination of lectures and small group work on guided worksheets.

Tentatively, we plan to cover:

- Topology foundations: CW complexes, delta and simplicial complexes. Cellular and simplicial approximation theorems. Higher homotopy groups. Hurewicz's theorem. Whitehead's theorem.
- Combinatorial topology: tools to understand the homotopy type of simplicial complexes, including Quillen's lemmas, link arguments, flow arguments, shellability, nerve lemmas, discrete Morse theory
- Applications, depending on time and student interest.

Prerequisites: Math 592 or equivalent.

IBL: Our course will use an Inquiry-Based Learning (IBL) format. For a portion of each class, students will work on exercises together in small groups. Development of collaboration and mathematical communication skills is an overarching goal of the course.

UNIVERSITY OF MICHIGAN

Math 797
Winter Term 2024

- Course Information
- Worksheets
- References
- Optional Reading
- Campus Resources

Class structure

- Begin class with 10-25 minutes of lecture.
- Students break into groups of 3-4. (Shuffled every $\sim$ 2 weeks.)
  Each group gets a chalkboard. Groups work on worksheets.
  I circulate and assist the groups.
- Worksheets have definitions, examples, problems,
  & bonus problems of increasing difficulty.
  Any technical questions are *highly* scaffolded.
- Groups work until every group finishes the non-bonus questions.
  Goal: 2-3 class periods per worksheet,
  enough bonus problems that no group runs out
- Each main problem assigned to one student to tex up in shared Overleaf document.
  I gave students feedback on solutions, and asked them to revise.
- Grade: 50% class participation, 50% worksheet solutions.

Sample Worksheet

1 PL Morse Theory

This worksheet is based (in part) on Bestvina's notes "PL Morse Theory". We will later prove the Theorem I as a special case of the "badness" technique (following Hatcher–Vogtmann) covered on upcoming Worksheet #19.

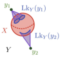
Theorem I. *Let Y be a simplicial complex, and $X \subseteq Y$ a subcomplex. Assume X and Y satisfy the following.*

(i) X is a full subcomplex of Y.

(ii) X is d-connected.

(iii) For all vertices y_1, y_2 in $Y \setminus X$, there is no edge $\{y_1, y_2\}$.

(iv) For all vertices $y \in Y \setminus X$, the link $\mathrm{Lk}_Y(y)$ is $(d-1)$-connected.



Then Y is d-connected.

Under the assumptions of the theorem, the complex Y is constructed from X by attaching a cone on $\mathrm{Lk}_Y(y) = \mathrm{Lk}_Y(y) \cap X$, with apex the vertex y, for each vertex $y \in Y \setminus X$.

Exercise 1. (Bonus) Prove Theorem I directly.

Exercise 2. (Bonus) Let $0 \leq k \leq n$ and let $\Delta^{n,k} = (\Delta^n)^{(k)}$. Consider the subcomplexes $\Delta^{n-1,k} \subseteq \Delta^{n,k}$ and $\mathrm{Lk}_{\Delta^{n,k}}(n)$. Use Theorem I and induction on n to conclude that $\Delta^{n,k}$ is $(k-1)$-connected.

Exercise 3. Prove: in a flag complex (such as the order complex of a poset), the star of a vertex is a full subcomplex.

Exercise 4. Let $0 \leq k \leq n$ be integers. Let $X^{n,k} = \mathrm{sd}\left((\Delta^n)^{(k)}\right)$ be the barycentric subdivision of the k-skeleton of an n-simplex. This is the order complex of the poset of subsets of $[n+1]$ of cardinality at most $(k+1)$, ordered by inclusion. As a warm-up application of Theorem I, we will re-prove that $X^{n,k}$ is $(k-1)$-connected. We proceed by induction on n, and induction on a filtration of $X^{n,k}$. Observe that Δ^0 is a point, hence is d-connected for all d. Fix $n \geq 1$ and assume by induction that the result holds for all $n' < n$ and all $0 \leq k \leq n'$.

(a) We define a filtration $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_{k+1} = X^{n,k}$ as follows. At each step, verify the description of the newly added vertices, and vertify that there are no edges between the newly added vertices.

- $X_0 = \mathrm{Star}_{X^{n,k}}(1)$
- X_1 is the full subcomplex on X_0 and all vertices corresponding to subsets of $[n+1]$ of cardinality 1 not contained in X_0, i.e, all vertices $\{a\}$ with $a \neq 1$.
- X_2 is the full subcomplex on X_1 and all subsets of cardinality 2 not contained in X_1, i.e., all vertices $\{a_1, a_2\}$ with $1 \neq a_1, a_2$.

  ⋮
- X_i is the full subcomplex on X_{i-1} and all subsets of $[n+1]$ of cardinality i not contained in X_{i-1}, i.e., vertices $\{a_1, \ldots, a_i\} \not\ni 1$.

  ⋮
- $X_{k+1} = X^{n,k}$

(b) For $i = 1, 2, \ldots k$, a vertex V in $X_i \setminus X_{i-1}$ is a subset of $[n+1]$ of cardinality i not containing the letter 1. Describe $\mathrm{Lk}_{X_i}(V)$, and verify that the vertex $V \cup \{1\}$ is a cone point of the link. (Why must we assume $i \leq k$?)

(c) The subspace X_0 is a cone (with vertex 1) and thus is contractible. Apply Theorem I inductively to deduce that $X_1, X_2, \ldots,$ and X_k are contractible.

(d) Verify that, if $n = k$, then $X_k = X^{n,k}$. Conclude the result in the case $n = k$.

(e) Suppose $n > k$. Let V be a vertex in $X_{k+1} \setminus X_k$. Show $\mathrm{Lk}_{X_{k+1}}(V) \cong X^{k,k-1}$.

(f) Use Theorem I and the inductive hypothesis on n to conclude Theorem II.

Our main application is a proof of the Solomon–Tits theorem in type A, a central result in the study of arithmetic groups. Let V be a finite-dimensional vector space over a field k. Recall from Worksheet #11 Section 1.5 that the *Tits building* (of type A), denoted $\mathcal{T}(V)$, is the order complex of the poset of proper, nonzero subspaces of V, ordered by inclusion.

Theorem II (Solomon–Tits Theorem, type A). *Let V be an n-dimensional vector space. The Tits building $\mathcal{T}(V)$ is $(n-3)$-connected.*

Since $\mathcal{T}(V)$ is not contractible, by Worksheet #7 Corollary V, Theorem II is equivalent to the statement that $\mathcal{T}(V)$ is a wedge of $(n-2)$-spheres.

Exercise 5. In this exercise we will prove Theorem II. The proof follows Section 5.1 of Bestvina's note. We proceed by induction on $\dim(V)$, and induction on a filtration of $\mathcal{T}(V)$.

(a) Verify that Theorem II holds when $\dim(V) = 1$.

(b) Let V be a k-vector space of dimension $n \geq 2$, and assume by induction the theorem holds for vector spaces of dimension $1, 2, \ldots, n-1$. Fix a line $L_0 \subseteq V$, and define a filtration $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_{n-1} = \mathcal{T}(V)$ as follows. Verify, at each step, the description of the newly added vertices, and vertify that there are no edges between the newly added vertices.

- $X_0 = \mathrm{Star}_{\mathcal{T}(V)}(L_0)$
- X_1 is the full subcomplex on X_0 and all lines $L \subsetneq V$ not in X_0. These are all lines satisfying $L \neq L_0$.
- X_2 is the full subcomplex on X_1 and all planes $P \subsetneq V$ not in X_0. These are all planes satisfying $P \not\supseteq L_0$.

  ⋮
- X_i is the full subcomplex on X_{i-1} and all subspaces $W \subsetneq V$ of dimension i not contained in X_0, that is, all i-dimensional subspaces W satisfying $W \not\supseteq L_0$.

  ⋮
- $X_{n-1} = \mathcal{T}(V)$

(c) For $i = 1, 2, \ldots n-2$, a vertex in $X_i \setminus X_{i-1}$ is a subspace W of dimension i not containing L. Describe $\mathrm{Lk}_{X_i}(W)$, and verify that the vertex $W \oplus L_0$ is a cone point of the link. (Where are we using the assumption that $i \leq n-2$?)

(d) The subspace X_0 is a cone (with vertex L_0) and thus is contractible. Apply Theorem I inductively to deduce that $X_1, X_2, \ldots,$ and X_{n-2} are contractible.

(e) For a vertex W in $\mathcal{T}(V) \setminus X_{n-2}$, show that $\mathrm{Lk}_{\mathcal{T}(V)}(W) = \mathcal{T}(W)$.

(f) Use Theorem I and the inductive hypothesis on n to conclude Theorem II.

Exercise 6. (Bonus) Using the proof of Exercise 5 and induction on $\dim(V)$, describe a basis for $\widetilde{H}_{n-2}(\mathcal{T}(V))$. In the case that k is a finite field, find a formula for the rank of this group.

Exercise 7. (Bonus) Let $V \cong k^{2n}$ be a symplectic vector space with symplectic form ω. The Tits building $\mathcal{T}^\omega(V)$ is the order complex of the poset of isotropic subspaces of V under inclusion. Adapt the proof of Exercise 5 to prove that $\mathcal{T}^\omega(V)$ is $(n-2)$-connected (and hence a wedge of $(n-1)$-spheres). *Hint*: See Section 5.2 of Bestvina's notes.

IBL Best Practices

IBL best practices:

- Have an explicit code of conduct for group work.
- Stress importance often.
- Seek student input.

From class webpage:

Class conduct: Class discussions and small group work are major components of this course. Students are expected to be active participants in the classroom, and are expected to conduct themselves with professionalism and respect for their classmates. Our goal is to create a supportive class environment where students are comfortable testing ideas, questioning each others' ideas, having their ideas challenged, and working together to reach a solution.

The student 'participation' grade is based on the following expectations. Students should ...

- attend class and participate in a group discussion
- present ideas and contribute to the discussion
- ensure their groupmates have equal opportunity to contribute
- make a genuine effort to engage with their groupmates' ideas
- treat their groupmates with patience and encouragement
- take responsibility for speaking up when they are confused
- take responsibility for ensuring their groupmates are included and are understanding the discussion.

Pros and cons of IBL format vs traditional lectures

- **Pro:** student feedback on IBL format uniformly positive!
- **Con:** cover *much* less material
- **Pro:** students engage longer and more actively with each topic
- **Pro:** teacher can assess student comprehension in real-time
- **Con:** for class sizes > 16, it is hard for one teacher to provide enough real-time support
- **Con:** may need to manage student interpersonal dynamics
- **Pro:** can have (class or individual) discussions with students about how to collaborate effectively and respectively, how to have confidence to ask questions, how to project encouragement, etc
- **Con:** lose control over pacing
- **Con:** lose control over which technical details students focus on
- **Con:** worksheets take time to prepare (offset by slowed pace ?)
- **Pro:** worksheets are useful for reading courses and PhD student training. Worksheets can be re-used if class is offered again.

Thank you!

Questions? Q&A period!