# Active learning in proof-based undergraduate math courses

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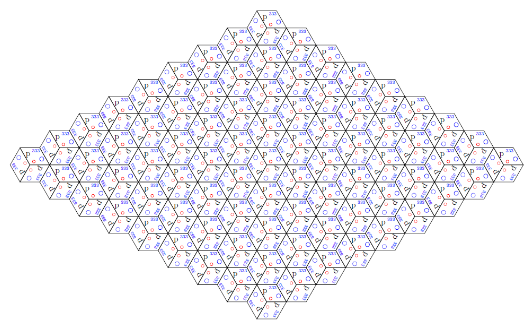

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## The challenge of teaching proof-based courses

Teaching 300 level proof-based math courses presents significant challenges:

*Introducing the language of abstract mathematics and methods of proof to students for the first time.*

The strongest students (for example, in honors sections) adapt easily and have strong performance.

From 2000 to 2015, the Rutgers math department had a rapidly increasing enrollment of students in 300 level proof-based math courses (mostly math majors and minors).

Many of these courses require cumulative skills.

Many students needed extra support.

## Active Learning comes into focus (2014)

A broad definition of active learning is that it involves introducing activities in the classroom that actively engage students in the process of learning rather than listening and note taking that are common to lecture style classes (Freeman et al., 2014)

# Why active learning?

A meta analysis of 225 studies (Freeman et al., 2014) examined examination scores (identical or equivalent examinations) and failure rates in STEM courses using traditional lecturing or active learning:

The active learning environment had higher assessment scores (on average a difference of about half a standard deviation).

The active learning environment had a lower failure rate (21.8% vs. 33.8% in traditional lectures).

## A CALL TO ACTION (2016)

Conference Board of the Mathematical Sciences (CBMS) Statement on Active Learning

'We call on institutions of higher education, mathematics departments and the mathematics faculty, public policy makers, and funding agencies to invest time and resources to ensure that effective active learning is incorporated into post secondary mathematics classrooms.'

https://www.cbmsweb.org/2016/07/active learning in post secondary mathematics education/ 4

This was echoed by an article in the notices of the AMS about active learning in mathematics:

http://www.ams.org/publications/journals/notices/201702/rnoti-p124.pdf

# Intro Math Reasoning

Math 300: Intro Math Reasoning is a required course for math majors, the first proof oriented class that students take.

The number of sections of Math 300 roughly tripled from 2000 to 2015.

26% DFW rate (11% failure rate)

The material in this course is a foundation for future courses (some of which are required courses for a math major).

Rutgers SAS requested that we take a different approach to reduce the DFW rate.

# Active learning initiative (2017)

In 2017, we began two concurrent initiatives in collaboration with the Rutgers Learning Centers:

- Introduction of active learning with Learning Assistants into certain sections of Math 300.

- A research study on the effects of active learning:

*Impact of active learning on course performance and self-reported learning gains in a proof-based mathematics course.*

Blackwell, S., Carbone, L., Katzen, S., Mejía Ramos, J. P., Ptak, C., Sandberg, A., & Seneres, A.

# Learning Assistants

The Learning Assistant Program is a national program developed at University of Colorado, Boulder.

Learning Assistants (LAs) are students who work with faculty to create collaborative learning environments.

Rutgers University has one of the largest Learning Assistant Programs in the country with over 200 LAs across campus.

Learning Assistants are extensively trained in pedagogy by the Learning Centers.

# Implementation of the Learning Assistants Program

The subject matter for Math 300: Intro Math Reasoning, is flexible (basic set theory, basic logic, basic number theory, relations and functions, countability).

Lecture notes may be adapted into workshop exercises for group work with relative ease.

As course head, I provided workshop material for all instructors based on my lecture notes.

In my sections, I allocate 15% of the final grade to course participation.

Student participation in workshops, and solving problems at the board in front of the class, count towards class participation.

# Implementation of the Learning Assistants Program

Workshops are held for 30-40 minutes per week (25% of class time) in around half the sections.

Tenured and tenure track faculty chose whether or not to work with an LA. Non tenured or tenure track faculty are required to work with LAs and run workshops.

SAS initially provided funding for LAs. Later, the math department shared the cost of LAs.

## A SAMPLE WORKSHOP QUESTION - LECTURE 2

The set of integers is denoted $\mathbb{Z}$. Let $u$ and $v$ be integers with $u \neq 0$. We say that $u$ *divides* $v$ if there exists an integer $c$ such that $v = uc$. This is equivalent to the statement that $\frac{v}{u} \in \mathbb{Z}$.

*Let $a, b$ be integers with $b \neq 0$. Suppose that the quadratic equation $x^2 + ax + b = 0$ has distinct solutions $z$ and $w$. If $z \in \mathbb{Z}$ and $w \in \mathbb{Z}$, prove that $z$ divides $b$ and $w$ divides $b$.*

*Solution.* Since $z, w$ are solutions, $x - z$, $x - w$ are factors of $x^2 + ax + b$. Thus we have

$$(x-z)(x-w) = x^2 - (w+z)x + wz = x^2 + ax + b.$$

Equating coefficients, we obtain $b = wz$ with $b \in \mathbb{Z}$ and $w, z \in \mathbb{Z}$. Note that since $b \neq 0$, we must have $w \neq 0$ and $z \neq 0$. Thus

$$\frac{b}{w} = z \in \mathbb{Z} \text{ and } \frac{b}{z} = w \in \mathbb{Z}.$$

Thus $z$ divides $b$ and $w$ divides $b$.

# The study

*'Impact of Active Learning on Course Performance and Self Reported Learning Gains in a Proof Based Mathematics Course'*

Blackwell, S., Carbone, L., Katzen, S., Mejía Ramos, J. P., Ptak, C., Sandberg, A., & Seneres, A.

We collected data in the form of pen+paper surveys of around 110 students over 3 years.

We checked there were no statistically significant differences between the control group and the study group with respect to:

SAT Math and Verbal scores, and Calc I and II grades.

We collected a 'pre' survey of student expectations.

# The surveys

Performance based pen+paper proof comprehension tests:

*Existence of infinitely many primes*

*Every third Fibonacci number is even*

Mejía Ramos, J. P., Lew, K., de la Torre, J., & Weber, K. (2017). *Developing and validating proof comprehension tests in undergraduate mathematics*, Research in Mathematics Education, 19(2), 130-146. https://doi.org/10.1080/14794802.2017.1325776

Available at

http://pcrg.gse.rutgers.edu/

- General idea/method understanding of the proofs

- Line by line (local) understanding of the proofs

Self reported learning gains:

*SALG-M (Student Assessment of their Learning Gains - Math)*

Laursen, Hassi, Kogan, Hunger, and Weston (2011)

# Results

No statistically significant difference in the distribution of course grades.

No statistically significant difference in the distribution of responses to questions related to cognitive gains.

Statistically significant differences in the distribution of responses to questions related to positive attitude, confidence, persistence and collaboration.

No statistically significant difference in the grades of the future course Math 311: Intro to Analysis.

Research shows that greater class time devoted to active learning (50-60% of class time) is needed to obtain statistically significant cognitive gains and an increase in course grades.

## UNMEASURABLE BENEFITS

Students slowly made friends and built a network with each other that extended outside of the classroom.

Working actively with students and LAs gives deeper insight into student progress and thinking.

Some students thrive in the active learning environment and even surprise themselves with their progress.

# Ongoing challenges

- Ensuring satisfactory implementation and organization of workshops.

- Suitable encouragement of joint work between students.

- Cooperation of instructors.

- Resistance from a small number of students.

# Collaborating with Mathematicians to Enhance Teaching (COMET)

NSF funded project focused on proof-based linear algebra Math 350.

Lisa Carbone, Rutgers Math Department,

Keith Weber, Rutgers Graduate School of Education,

Estrella Johnson, Virginia Tech Math Department,

Tim Fukawa-Connelly, Temple University College of Education.

Research Assistant: Hamidreza Mahmoudian, Rutgers Math Dept

Math 300: Intro Math Reasoning is a prerequisite for this course.

## ABSTRACT LINEAR ALGEBRA

Main challenges of teaching proof-based linear algebra:

- Guiding students who are novices at writing proofs into the abstract and axiomatic world of vectors spaces.

- Introducing questions and class activities that involve

1. Definitions,
2. Deep concepts,
3. Reasoning,
4. Following steps in proofs,
5. Applying hypotheses of theorems.

- Many students still require review of basic techniques of proof.

## OBJECTIVES

Work with professors of mathematics to develop and test various active learning strategies in their proof-based linear algebra course Math 350.

Interview and video record certain classes of mathematics professors who are collaborating on this project.

Weekly meetings between the PI's and mathematics professors.

Develop questions and activities for students and discuss their implementation.

Analyze the reported outcomes by faculty of the implementation of these activities in class.

# Implementation of suggested activities

All instructors had previously used their own (often effective) methods for engaging students.

However, such methods may not have been systematic or given concrete feedback to instructors from all students.

The syllabus has a number of high level required theorems and proofs.

Given the extensive syllabus, there is a lack of class time to experiment with interactive exercises.

## Sample questions

**Reasoning**

Let $V$ be the vector space of infinitely differentiable real valued functions over $\mathbb{R}$. Let $T$ be the linear transformation on $V$ given by $T(f(x)) = f'(x)$.

Claim: Every real number is an eigenvalue of $T$.

Justification: Let $\lambda \in \mathbb{R}$. If $f(x) = e^{\lambda x}$ then $T(f(x)) = \lambda e^{\lambda x} = \lambda f(x)$.

Is the justification correct?

**Theorem conditions**

Let $P^2(\mathbb{R})$ be the vector space of polynomials of degree less than or equal to 2 over $\mathbb{R}$. Suppose that $T$ is a diagonalizable linear operator on $P^2(\mathbb{R})$. Let $\beta$ be the standard basis $\{1, x, x^2\}$. Let $[T]_\beta$ be the matrix of $T$ with respect to this basis. Is the following statement correct?

*Since T is diagonalizable, $[T]_\beta$ is a diagonal matrix.*

# Ongoing research

Collaboration with mathematics professors will be repeated next Fall with the benefit of experience from this semester.

The remainder of the grant funded period will involve analyzing data collected.

We hope to develop a collection of useful activities to engage students and give feedback to instructors.

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