Implementing Projects in Abstract Algebra

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A few words about myself…

- I'm a Midwesterner!
  - Ravenna High School in 1983
  - BA, Kent State University in 1987
  - PhD, Univ. of Illinois-Urbana in 1994

- I've taught at a range of institutions:
  - U.S. Air Force Academy (1994-97)
  - Kenyon College (in my 27th year)
  - Carnegie Mellon University (sabbatical)
  - Harvard University (several summers)

Quick Overview:

- Context and Motivation
- Sample Project 1
- Sample Project 2
- Sample Project 3
- Philosophy behind these projects
- A Brief mention of my use of projects in Abstract II

The bulk of my talk!

Some Background:

Kenyon offers a year-long sequence in Abstract Algebra

Algebra I: Groups

enrollment: ~20 students

(mostly students interested in math, physics, or CS and future teachers)

Abstract II: Rings and Fields (with a focus on Fields)

Or Rings and Fields (with a focus on Algebraic Geometry)

enrollment: ~10 students

(mostly students interested in graduate study in math, physics or CS)

What does the research say about the difficulty of Abstract Algebra?

"Students often have difficulties relating activities such as creating tables and bijections (Weber and Larsen, 2004), performing calculations (Hazzan, 1999) and finding the order of elements to the group concepts pertaining to these activities. (Leron et al., 1995) Students will commonly use algorithms to solve problems without correlating them with their theoretical knowledge."

-- J. Lewis, "An Analysis of Students' Difficulties in Learning Group Theory," 2013.

What does the research say about the difficulty of Abstract Algebra?

"most college students do not succeed in understanding the concept of a quotient group. Indeed, major difficulties for students seem to begin appearing with the introduction of cosets."

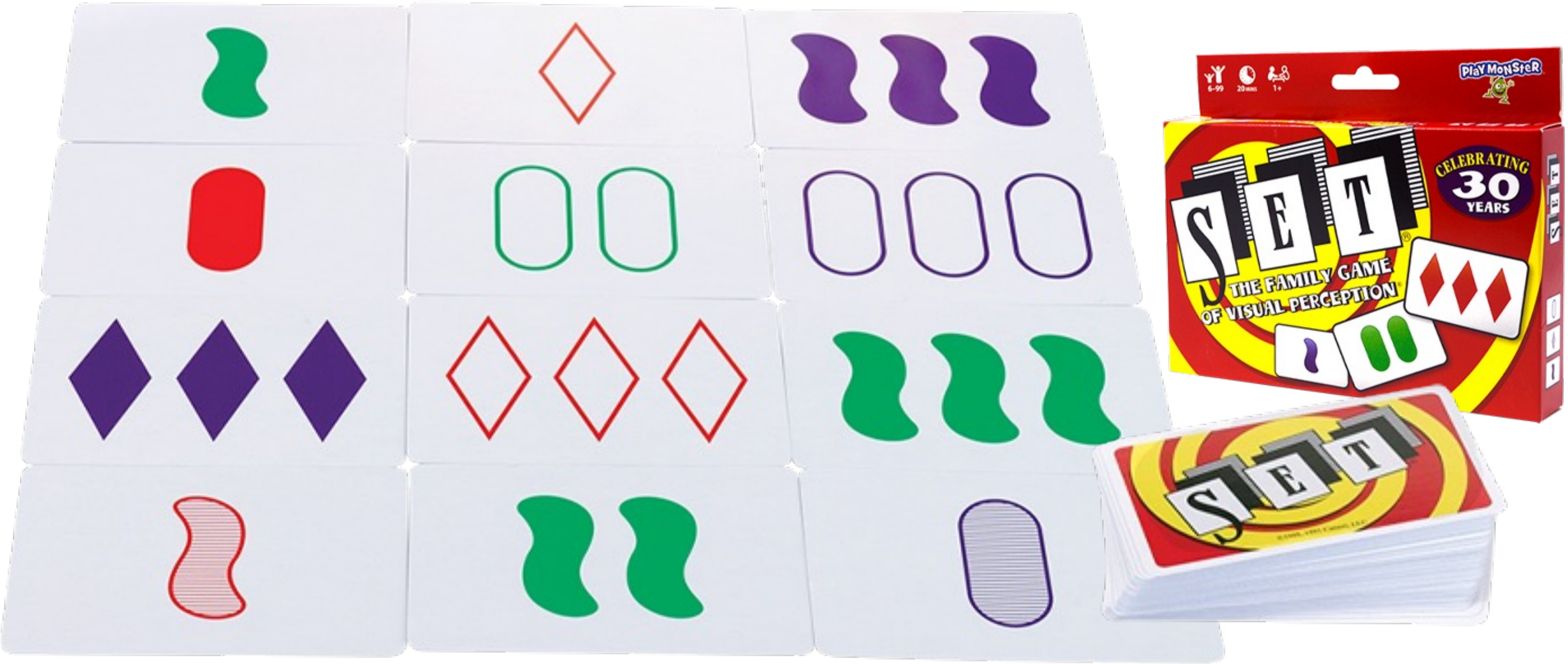
-- E. Dubinsky, J. Dautermann, U. Leron, R. Zazkis
"On Learning Fundamental Concepts of Group Theory," 1994

My Initial Motivation for Creating Projects:

- Create opportunities for students to gain intuition about abstract objects
- Create environments in which students can experiment and discover new ideas on their own (although not entirely on their own; there are guiding exercises)
- To reveal the purpose of abstract algebra; by applying an algebraic framework to a "real world" problem, students can see the power of the toolbox.

Example Project 1:

“Product-free Sets in the Card Game SET”

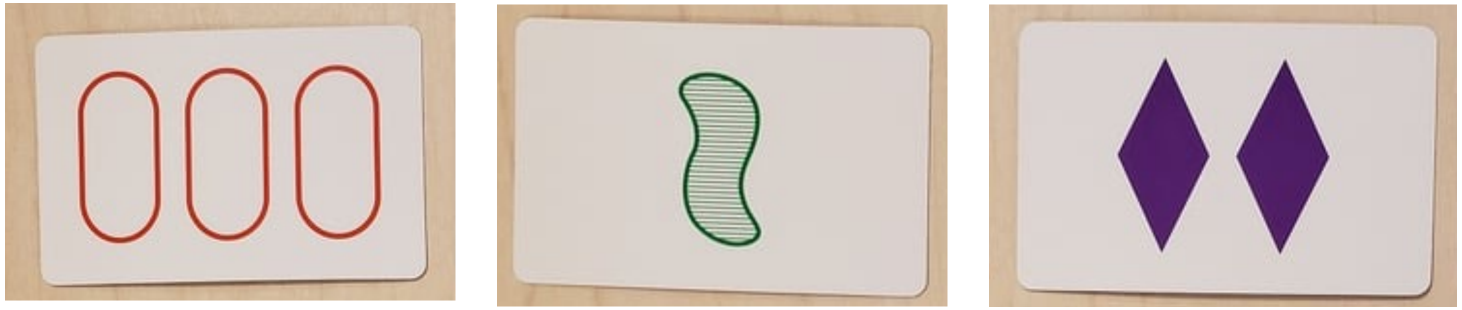


Object of the Game:

Identify a “SET” among an array of 12 cards laid on the table.

Each card exhibits four characteristics:

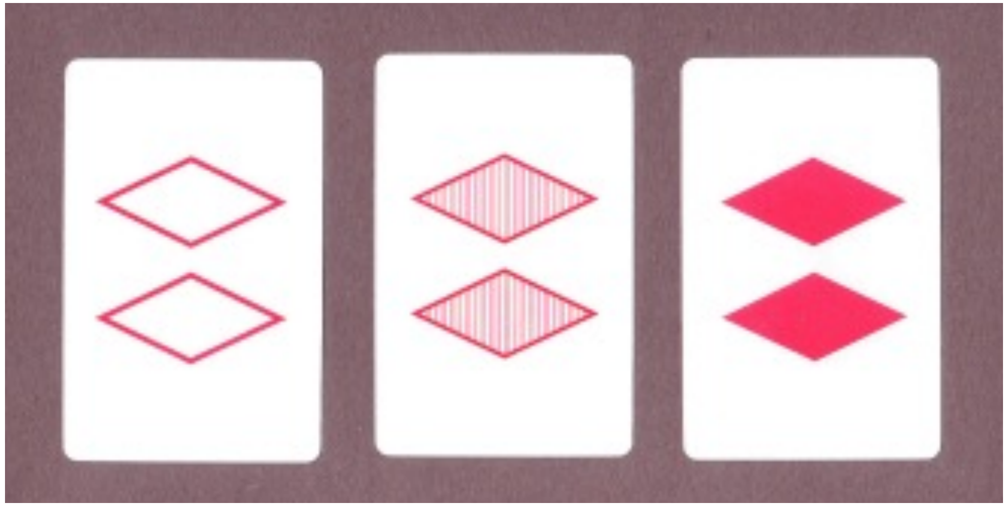
1. **Symbol:** Each card contains **ovals**, **squiggles**, or **diamonds**
2. **Color:** The color of the symbols is **red**, **green**, or **purple**
3. **Number:** There are **1**, **2**, or **3** symbols per card
4. **Filling:** The symbols are **filled-in solid**, **unfilled**, or **striped**



There are a total of $3^4 = 81$ cards in a SET deck.

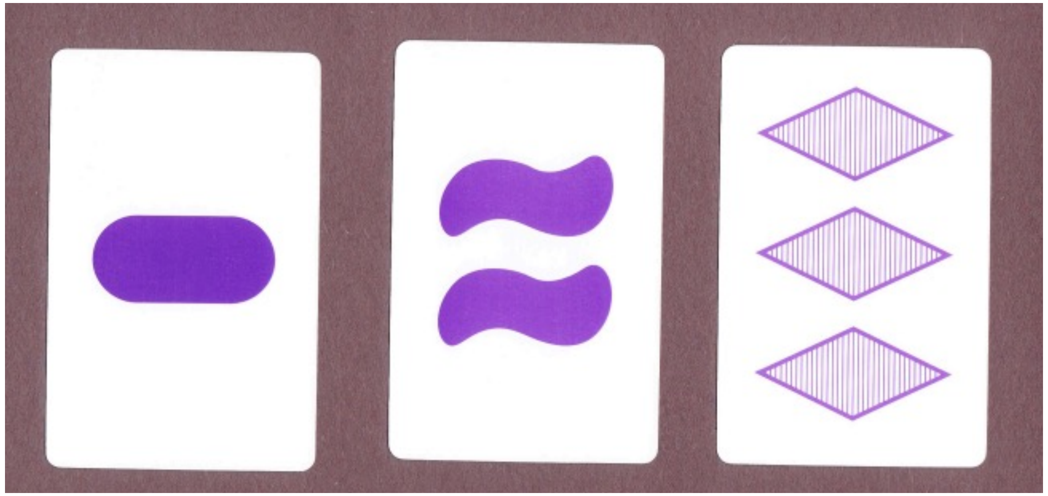
Three cards make up a **SET** if, for each of the four characteristics, all three cards either share the characteristic or each is different.

Example of a Set



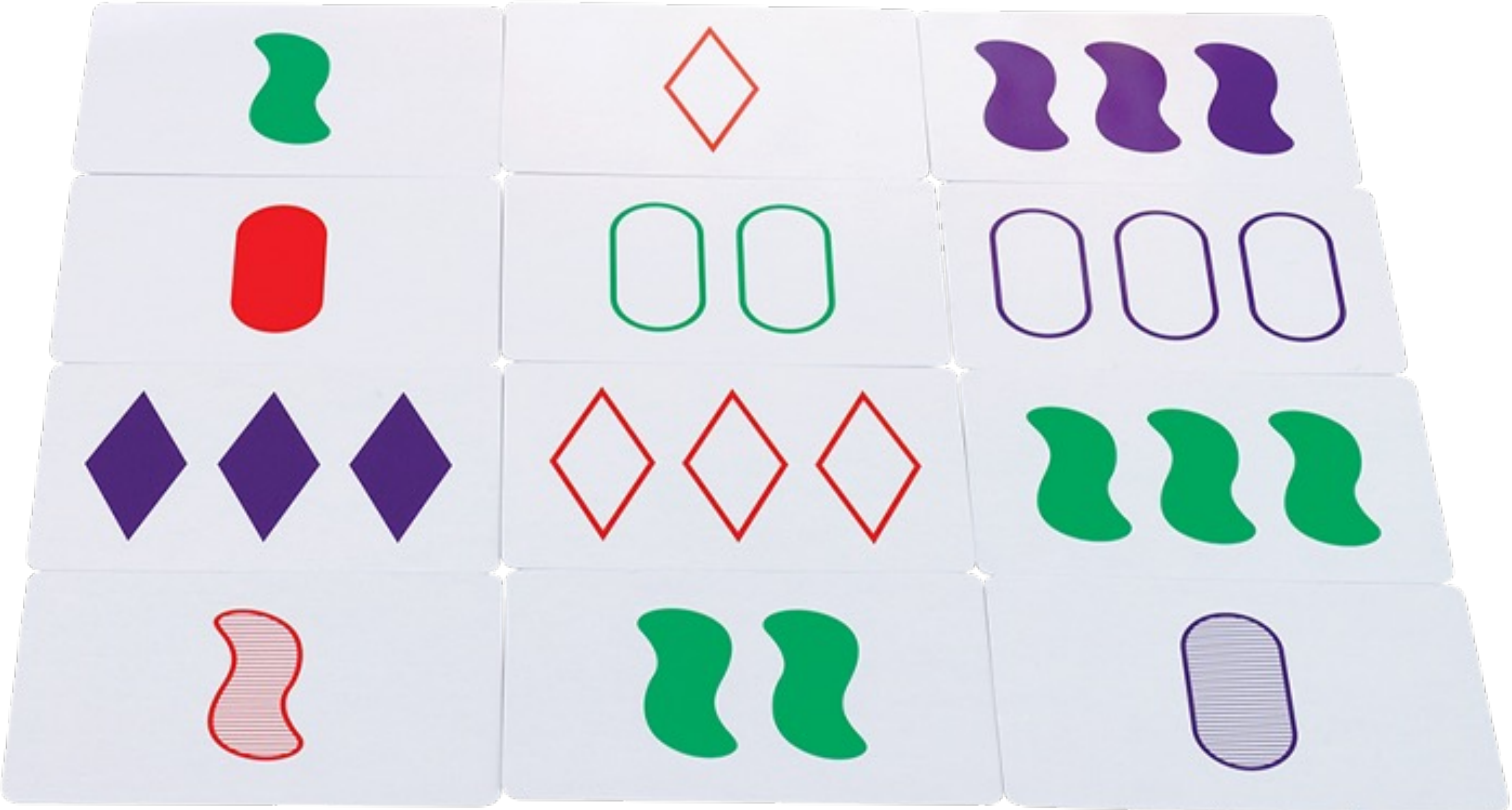
All share color, number and symbol; all differ in filling

Example of a non-Set

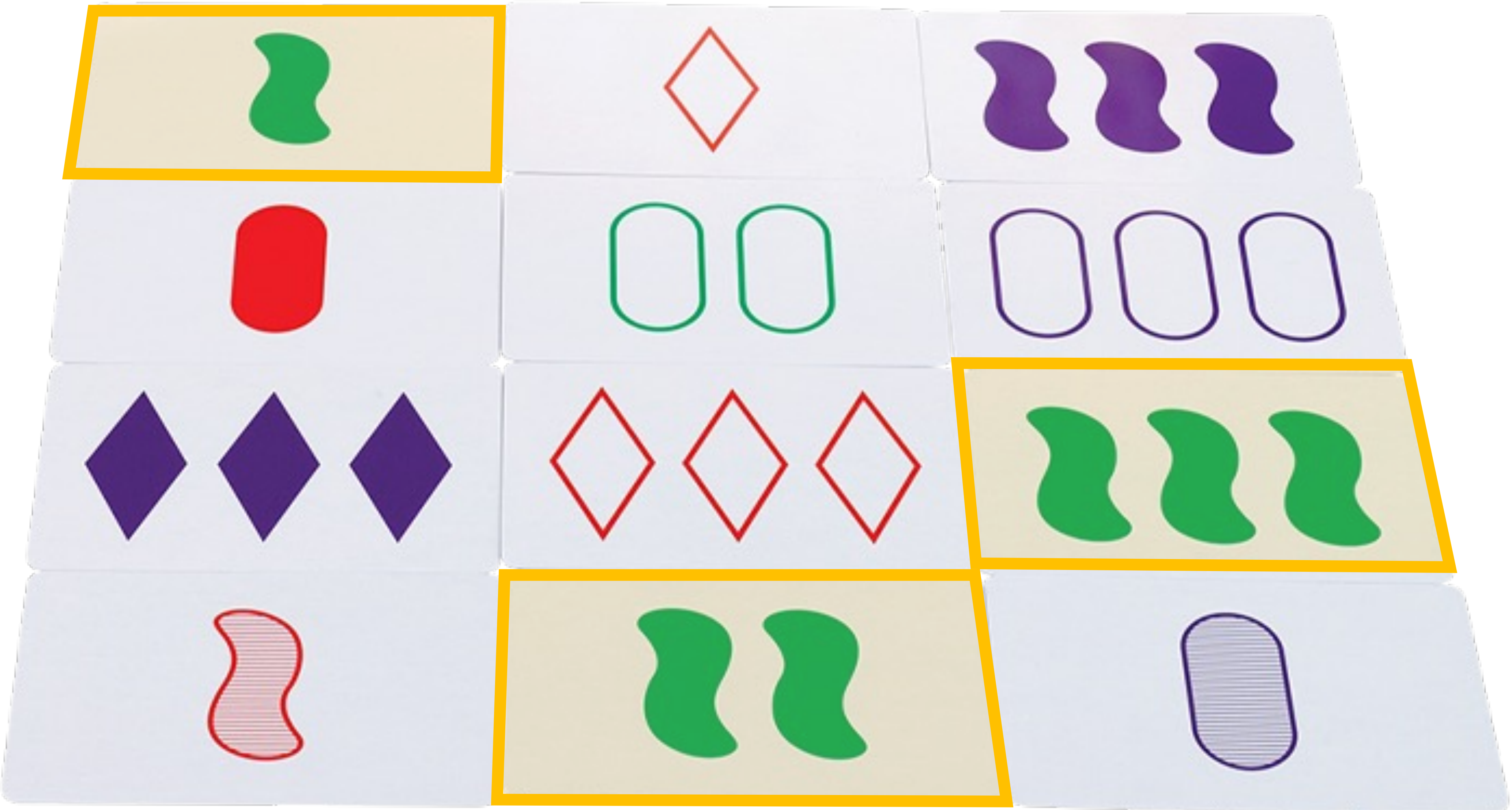


Only 2 of the 3 share the solid filling

Let’s Play!

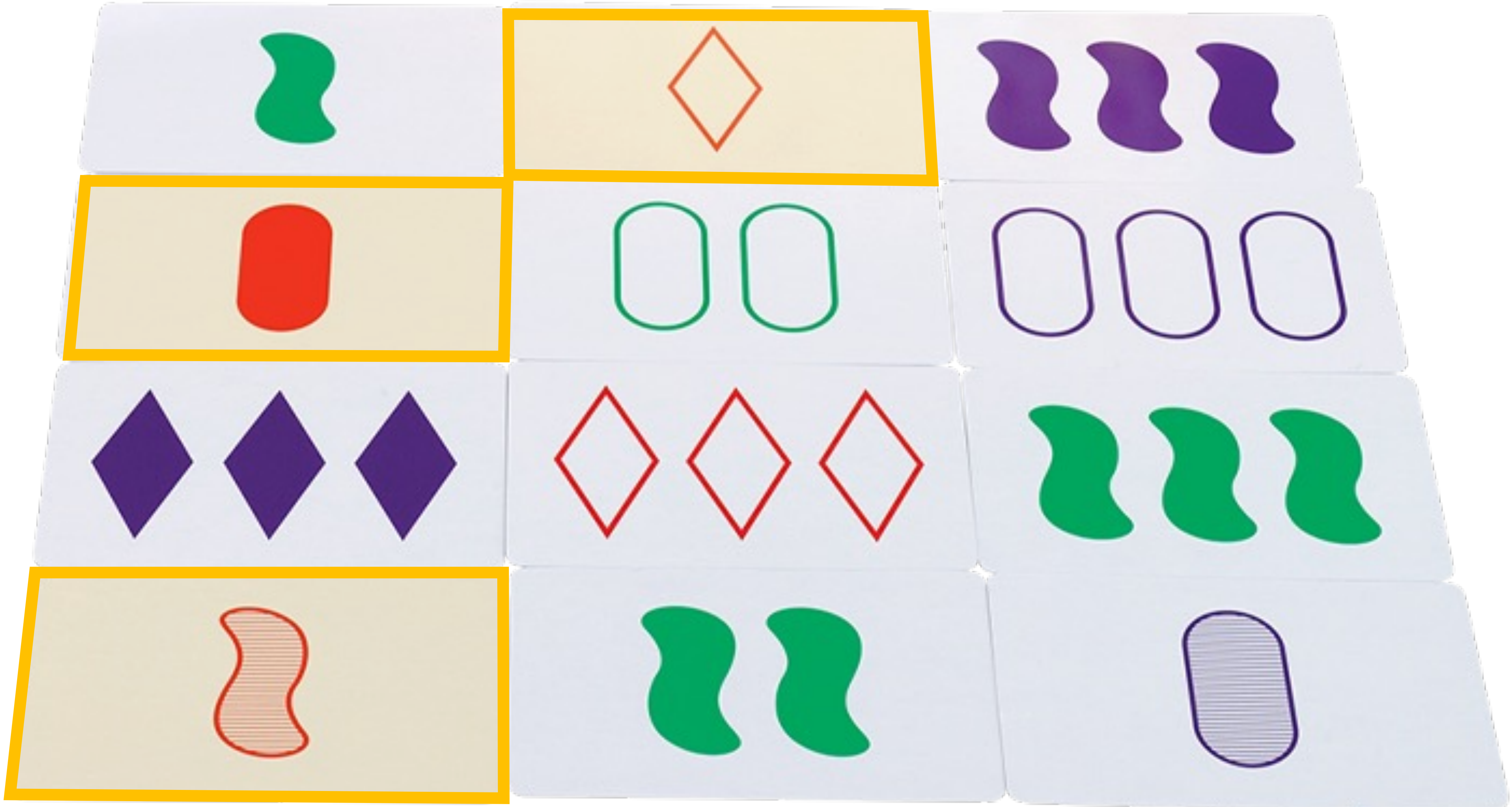


SET!



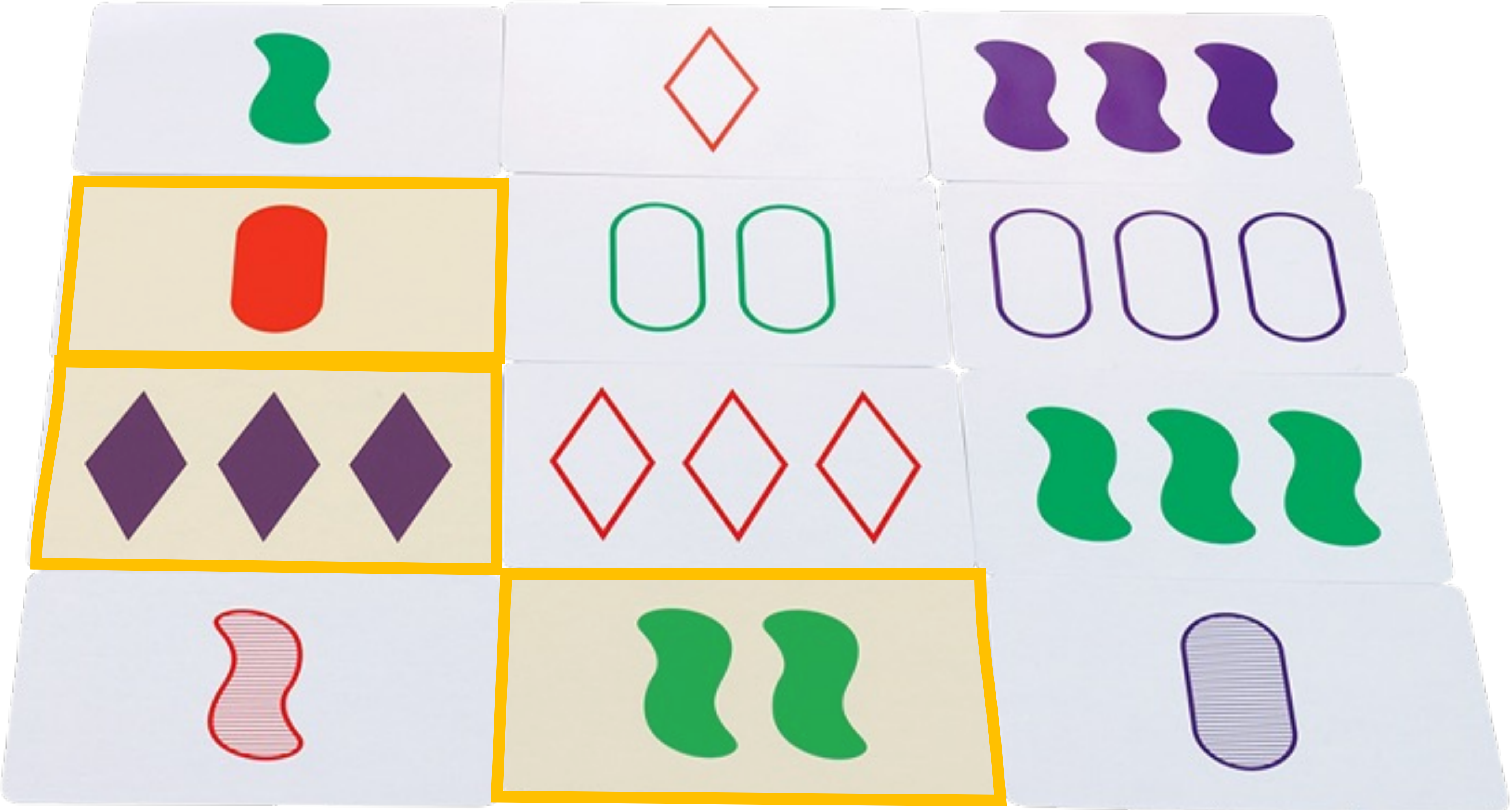
Color, symbol & shading the same; number different

SET!



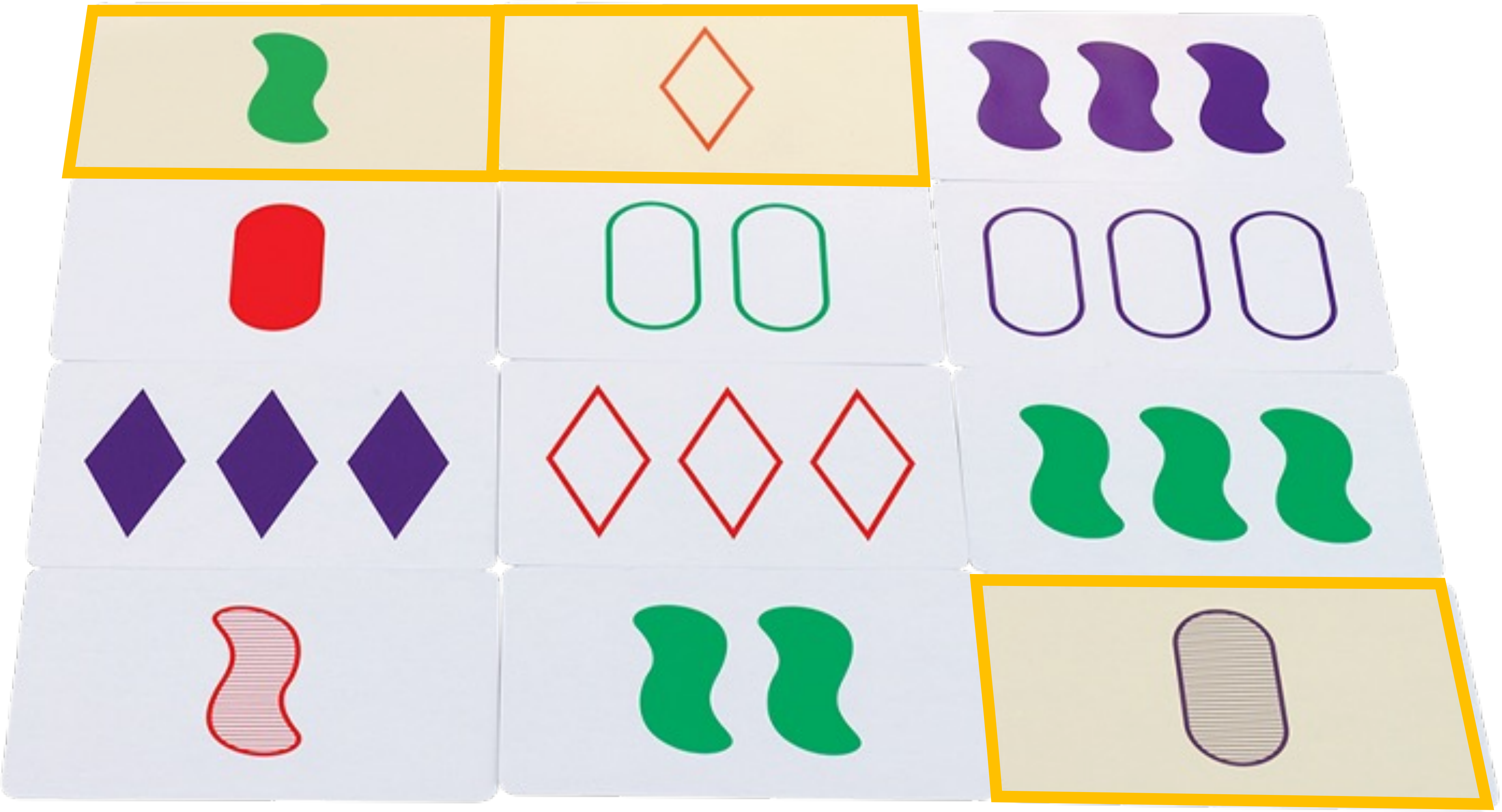
Color & number the same; symbol & shading different

SET!



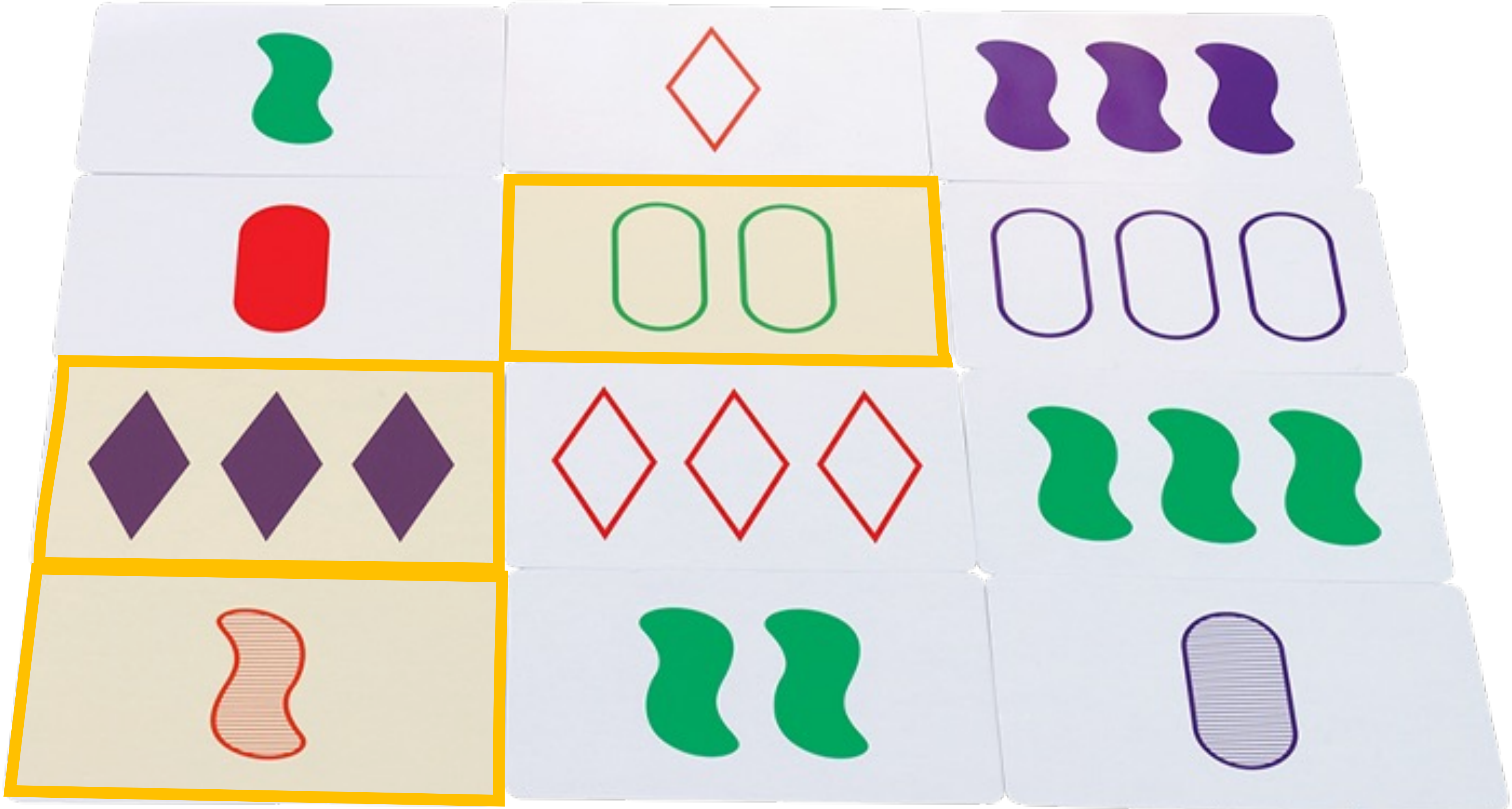
Shading the same; symbol, color & number different

SET!



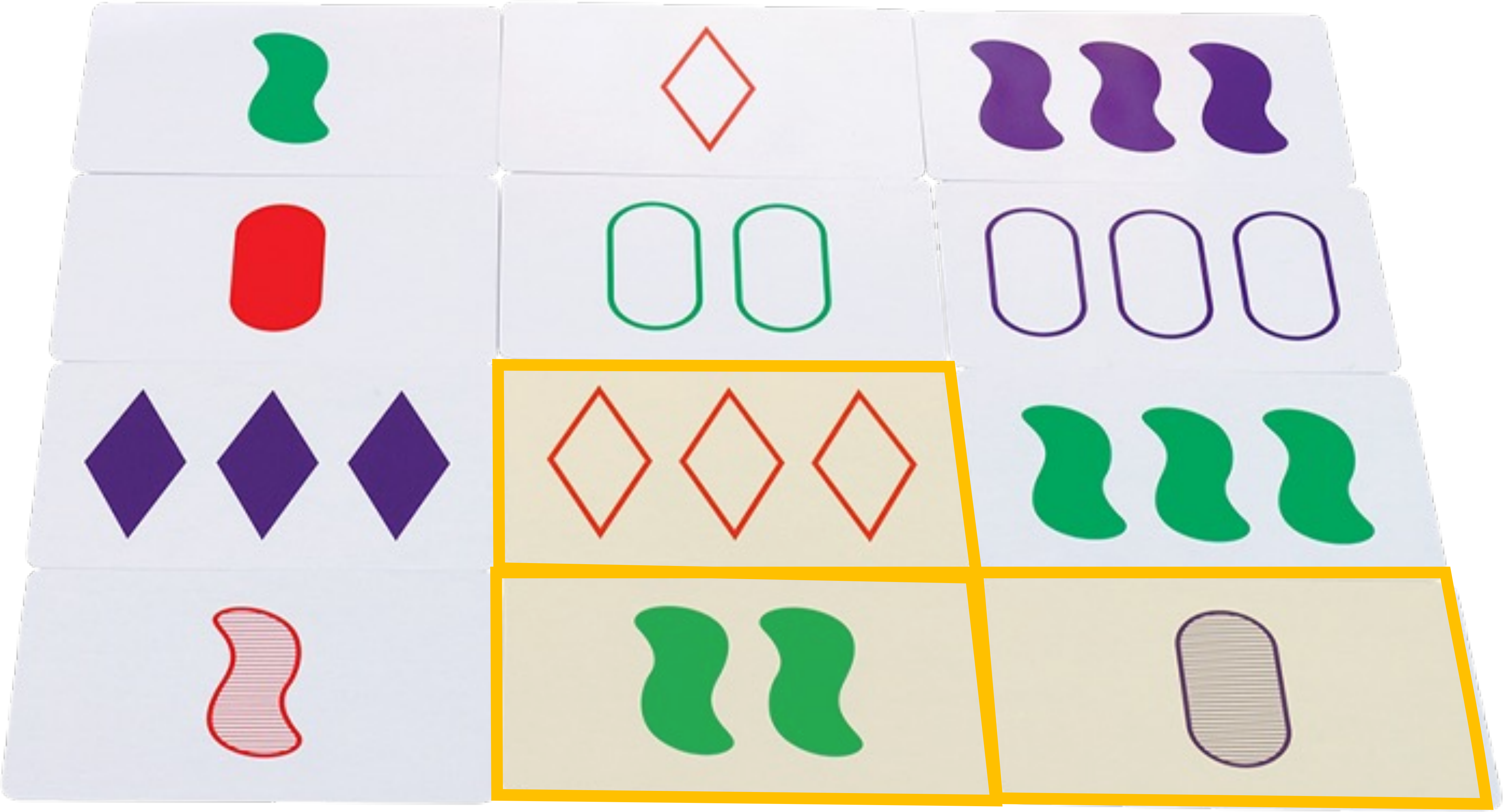
Number the same; symbol, color & shading different

SET!



symbol, color, number & shading all differ

SET!



symbol, color, number & shading all differ

It's possible to have no SET among the 12 cards

According to the instructions, among an initial layout of 12 cards, $P(\text{no SET}) = 1/33.$

If there is no SET among the 12 cards on the table, you lay out three more cards.

According to the instructions, among an initial layout of 15 cards, $P(\text{no SET}) = 1/2500.$

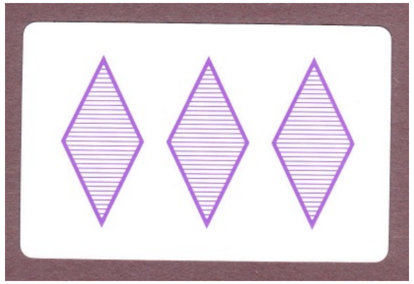
Question: What is the maximum possible number of cards you can have producing no SET?

The Mathematical Framework

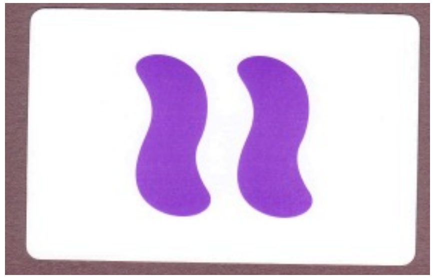
Associate each card in the SET® deck with an element in the set

$$D = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$$

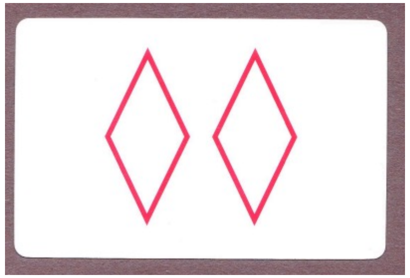
Labeling	Symbol	Color	Number	Filling
1	Oval	Red	One	Solid
2	Squiggle	Green	Two	Striped
0	Diamond	Purple	Three	Outlined



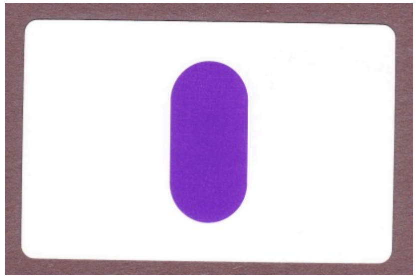
(0, 0, 0, 2)



(2, 0, 2, 1)



(0, 1, 2, 0)



(1, 0, 1, 1)

Defining the Product of Two Cards

Given $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ in D, define

$$xy = \left(\frac{x_1 + y_1}{2} \bmod 3,\ \frac{x_2 + y_2}{2} \bmod 3,\ \frac{x_3 + y_3}{2} \bmod 3,\ \frac{x_4 + y_4}{2} \bmod 3\right)$$

The Project:

Exercise 1. Suppose $x = (1, 1, 1, 1)$, $y = (0, 0, 0, 0)$, $z = (1, 2, 2, 0)$ and $w = (2, 2, 1, 1)$. Compute each of the products xy, xz, xw, and zw (symbolically), and then sketch the corresponding card to determine how this multiplication translates into cards.

The Project (continued):

Exercise 2 Once you understand what it means to multiply two cards together, you might wonder if D is a group under this multiplication. Let's explore this question next. Answer each of the following questions, providing proofs of each of your claims.

a) Is the multiplication on D associative?
b) Is the multiplication commutative?
c) Does D contain an identity element?
d) Do inverses exist?

Exercises 3-5 require students to prove several properties hold true, including the cancellation laws. Students are also asked to translate the meaning of these properties into statements about cards.

Exercises 6-8 introduce a couple definitions:

- A ***D*-set** is defined to be a subset $S \subseteq D$ of the form $S = \{x, y, xy\}$, where $x, y \in D$.

  So D-sets correspond to SETs in the card game.

- A subset $U \subseteq D$ is **product-free** if $xy \notin U$ whenever $x, y \in U$.

  Product-free subsets of D translate into collections of cards that fail to contain a SET

  Our original question becomes:

  What is the largest product-free subset of D?

Then students are asked to prove that if a subset U is product-free, then $xU = \{xu \mid u \in U\}$ is product-free, as well.

The last couple of exercises get the students thinking about the size, $\alpha(D)$, of the largest product-free subset of D…

Exercise 9. Prove that if $S \subseteq D$ is a product-free set and $x \in S$, then $S \cap xS = \{x\}$. Conclude that any product-free set can contain at most 41 elements. (So any collection of 42 cards must contain a SET.)

Exercise 10 (An Optional Challenge). The size of the largest known collection of cards containing no Set is 20. If we let $\alpha(D)$ denote the size of the largest product-free subset of D, then this fact together with the result of the previous exercise yields $20 \leq \alpha(D) \leq 41$. Can you tighten the upper bound on $\alpha(D)$ using the framework introduced in this project? That is, can you find an upper bound that is less than 41, or optimally, can you prove that $\alpha(D)$=20?

Goals of the SET Project:

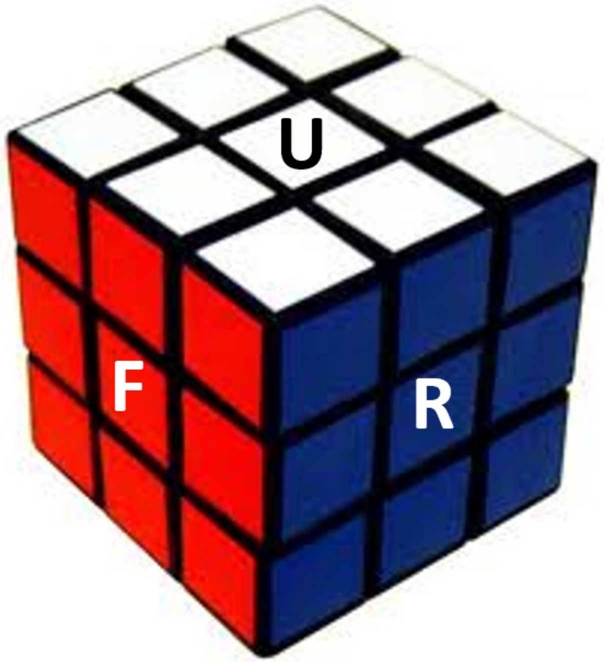
- Provide a nonstandard example of an algebraic structure, encouraging students to make conceptual connections
- Illustrate the use of a Cartesian product
  (Many students equate the Cartesian product with subsets of $\mathbb{R}^d$)
- Demonstrate that meaningful binary structures can fail to satisfy basic properties (associativity!)
- Give students practice working with coset-like objects xS, paving the way for future content
- Reveal the purpose of abstract algebra by applying an algebraic framework to a "real world" problem

Example Project 2:

“Exploring Rubik’s Cube with GAP”

Explores the transformation group of the Rubik’s cube using GAP (Groups, Algorithms and Programming) – a program for computational discrete algebra

Start with a Labeling of the Cube:



The labeling will allow us to describe motions on the Rubik's Cube in terms of permutations on the set of labels:

$$\{1, 2, 3, \ldots, 48\}.$$

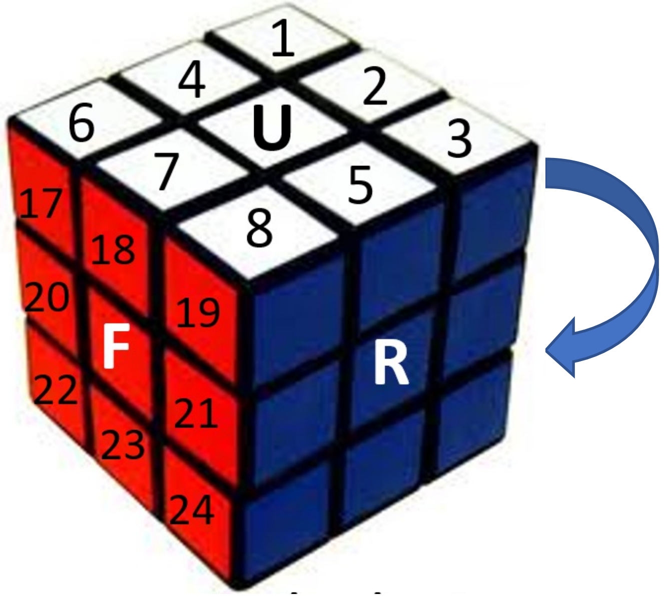
			1	2	3						
			4	**Up**	5						
			6	7	8						
9	10	11	17	18	19	25	26	27	33	34	35
12	**Left**	13	20	**Front**	21	28	**Right**	29	36	**Back**	37
14	15	16	22	23	24	30	31	32	38	39	40
			41	42	43						
			44	**Down**	45						
			46	47	48						

Exercise 1: Consider the motion described by:

$$U := (1, 3, 8, 6)\,(2, 5, 7, 4)\,(9, 33, 25, 17)\,(10, 34, 26, 18)$$
$$(11, 35, 27, 19)\,(12, 36, 28, 20)\,(13, 37, 29, 21)\,(14, 38, 30, 22)$$
$$(15, 39, 31, 23)\,(16, 40, 32, 24)\,(41, 46, 48, 43)(42, 44, 47, 45)$$

What motion does U represent?

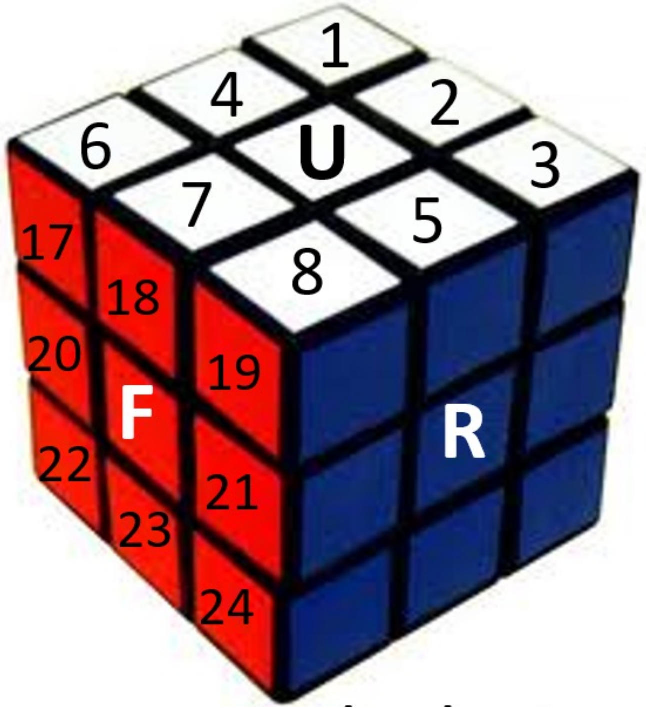
			1	2	3						
			4	**Up**	5						
			6	7	8						
9	10	11	17	18	19	25	26	27	33	34	35
12	**Left**	13	20	**Front**	21	28	**Right**	29	36	**Back**	37
14	15	16	22	23	24	30	31	32	38	39	40
			41	42	43						
			44	**Down**	45						
			46	47	48						



Answer: A clockwise 90° rotation of the entire cube about the vertical axis

Exercise 2: Suppose F denotes a clockwise turn of the front face of the cube. Describe F as a permutation.

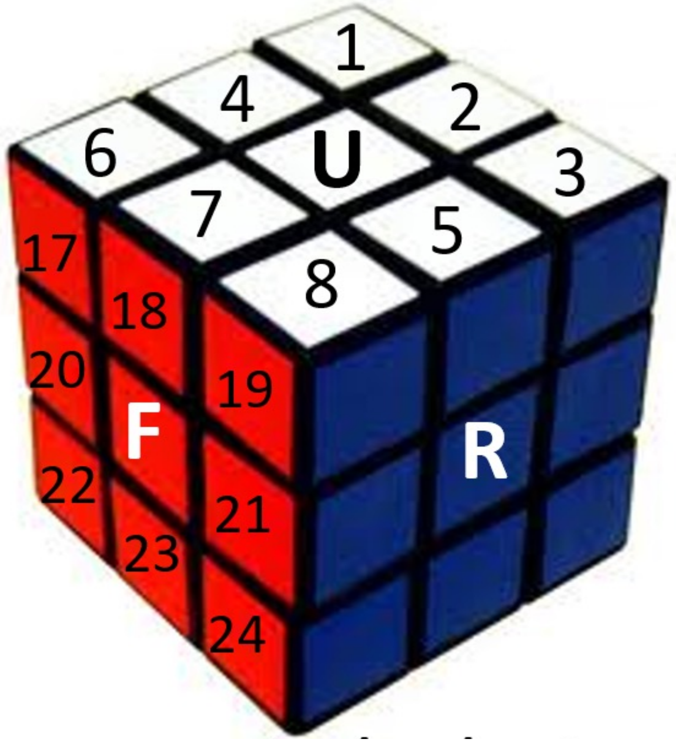
			1	2	3						
			4	**Up**	5						
			6	7	8						
9	10	11	17	18	19	25	26	27	33	34	35
12	**Left**	13	20	**Front**	21	28	**Right**	29	36	**Back**	37
14	15	16	22	23	24	30	31	32	38	39	40
			41	42	43						
			44	**Down**	45						
			46	47	48						



F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11)

Exercise 3: Determine a “natural” set of generators of the group of transformations of the Cube. That is, find a set of motions that will enable you to create any attainable configuration of the Cube. Then define (in **GAP**) each motion as a product of disjoint cycles.

F, B, U, D, L, R



Before continuing, define a subgroup $H \leq S_{48}$ to be the group generated by the set of generators you defined in Exercise 3.

```
gap> G:= SymmetricGroup(48);
gap> H:= Subgroup(G, [U,D,R,L,F,B]);
```

Exercise 4: Next we want to use **GAP** to explore what happens as a result of repeatedly applying a particular sequence of motions. Let F denote a clockwise turn of the front face, and R a clockwise turn of the right face. Evaluate the following permutations:

a) $\{R^2F^2, (R^2F^2)^2, (R^2F^2)^3, \ldots\}$

(Can you find any "strategic maneuvers" among this list? What is the order of R^2F^2?)

$(R^2F^2)^3 = (21, 36)(23, 39)(28, 29)(42, 47)$

b) $\{RFR^{-1}F^{-1}, (RFR^{-1}F^{-1})^2, (RFR^{-1}F^{-1})^3, \ldots\}$

(What is the order of $RFR^{-1}F^{-1}$?)

$(RFR^{-1}F^{-1})^3 = (6, 19)(8, 17)(11, 25)(24, 38)(30, 48)(32, 43)$

Exercise 5: When the original Rubik's Cube came out, the Ideal Toy Company stated on its package that "there were more than three billion possible states the cube could attain." In his book, *Innumeracy*, J.A. Paulos described this claim as "analogous to McDonald's proudly announcing that they've sold more than 120 hamburgers." Use **GAP**'s "Size" command to find the actual size of the Cube's group of transformations.

```
gap> G:= SymmetricGroup(48);
gap> H:= Subgroup(G, [U, D, R, L, F, B]);
gap> Size(H);
```

43,252,003,274,489,856,000 elements!

Exercise 6:

a) Next consider what happens when you repeatedly apply a cw turn of the right face followed by a cw turn of the front face. That is, evaluate the permutations $\{RF, (RF)^2, (RF)^3, \ldots\}$. What is the order of the cyclic subgroup generated by RF?

b) Letting B represent a clockwise turn of the back face and L a clockwise rotation of the left face, what is the order of the subgroup generated by RFL? What is the order of the subgroup generated by RFLB?

c) Choose your own sequence of motions to explore. Do you think you can find a single sequence that will generate the entire transformation group of the Cube? (That is, do you think the group of motions on the cube is cyclic?)

Exercise 7: Finally we want to examine our initial set of generators. Do you think that the set you defined in Exercise 3 is the smallest? Explore this idea. See if you can find a smaller set of generators. (Note: In general, it is very difficult to find a minimal set of generators of a group. Nonetheless, **GAP** gives us the power to experiment with such questions.)

With some help from GAP, one can show that F can be expressed as:
U^2 B^3 R^2 B^3 D^2 B^3 D L^3 B L^3 D^3 R U B U^3 R^3 B U B^3 R U B R U^3 R^3 U^3 B^2 U D^3 L^3 B L D B^3 L^3 B^3 L U^3 B^2 U^3 B^2 U^3 R^3 U R B U^2 B U B D B^3 U^3 B D^3 B^2 U B U^3 B^3 U^2 B^2 U L U^3 L^3 B^2 U^3 B^3 R B R^3 U B U B^3 R^3 U^3 R U B U^3 B^3 U^3 B^3 R B R^3 B U B^2 R^3 U^3 R U B^3 D B^3 U^3 B D^3 B^3 U^3 B^3 U

Goals of the Project:

- To build a better understanding of:
  -- permutations and permutation groups
  -- groups defined by generators
  -- cyclic groups
  -- the order of an element
- To give students an opportunity and the means to discover patterns, build intuition, and make conjectures.
- To reveal the purpose of abstract algebra by applying an algebraic framework to a "real world" problem

Example Project 3:
"Conjugation in Permutation Groups"

Explores the relationship between the cycle structure of a permutation and cycle structure of its conjugate; Revisits permutations of the Rubik's cube.

Exercise 1. Let $G = S_{10}$ and h1, h2, h3, h4, and h5 be as defined by:

gap> h1: = (1, 2, 3);

gap> h2: = (2, 3)(5, 4, 7);

gap> h3: = (1, 2, 3)(8, 9, 10);

gap> h4: = (1, 2, 4)(5, 9);

gap> h5: = (4, 6, 7, 9);

Complete the tables given below.

g1	(1, 2, 3, 4)(7, 8, 9)
$h1*g1*h1^{-1}$	(1, 2, 4, 3)(7, 8, 9)
$h2*g1*h2^{-1}$	(1, 3, 2, 5)(4, 8, 9)
$h2*g1*h2^{-1}$	(1, 2, 4, 3)(7, 10, 8)
$h2*g1*h2^{-1}$	(1, 3, 2, 4)(5, 7, 8)
$h2*g1*h2^{-1}$	(1, 2, 3, 9)(6, 8, 7)

g2	(1, 4, 3, 2)(5, 7, 8)
$h1*g2*h1^{-1}$	(1, 3, 4, 2)(5, 7, 8)
$h2*g2*h2^{-1}$	(1, 5, 2, 3)(4, 8, 7)
$h2*g2*h2^{-1}$	(1, 3, 4, 2)(5, 7, 10)
$h2*g2*h2^{-1}$	(1, 4, 2, 3)(7, 8, 9)
$h2*g2*h2^{-1}$	(1, 9, 3, 2)(5, 6, 8)

Exercise 1. Let $G = S_{10}$ and h1, h2, h3, h4, and h5 be as defined by:

gap> h1: = (1, 2, 3);

gap> h2: = (2, 3)(5, 4, 7);

gap> h3: = (1, 2, 3)(8, 9, 10);

gap> h4: = (1, 2, 4)(5, 9);

gap> h5: = (4, 6, 7, 9);

Complete the tables given below.

g3	(1, 3)(7, 8, 9)
$h1*g1*h1^{-1}$	(2, 3)(7, 8, 9)
$h2*g1*h2^{-1}$	(1, 2)(4, 8, 9)
$h2*g1*h2^{-1}$	(2, 3)(7, 10, 8)
$h2*g1*h2^{-1}$	(3, 4)(5, 7, 8)
$h2*g1*h2^{-1}$	(1 3)(6, 8, 7)

g4	(1, 7)
$h1*g2*h1^{-1}$	(3, 7)
$h2*g2*h2^{-1}$	(1, 4)
$h2*g2*h2^{-1}$	(3, 7)
$h2*g2*h2^{-1}$	(4, 7)
$h2*g2*h2^{-1}$	(1, 6)

Exercise 2. GAP will actually allow you to look at all of the elements in S_{10} that are conjugate to any given element. For example, to get the list of conjugates of *g5* you execute the following. (Be patient! This computation will likely take a while.)

```
gap> g5 := (1, 2, 5, 6);
gap> c:= ConjugacyClass(G, g5);
gap> Elements(c);
```

What does the output of this code seem to indicate? What can you say about the cycle structure of *g5* compared to the cycle structure of each of its conjugates?

Exercise 3. Use **GAP**'s "Size" command to determine whether or not every four cycle in S_{10} is conjugate to *g5*. Explain your reasoning.

Size(c) = 1260

The number of distinct 4-cycles in S_{10} is $(10)(9)(8)(7)/4 = 1260$, so yes, every 4-cycle in S_{10} is conjugate to g5.

Exercise 4. To summarize your findings, provide a conjecture about the relationship between a permutation $g \in S_{10}$ and any conjugate $h*g*h^{-1}$, where $h \in S_{10}$. *Your homework assignment for next lesson is to prove (or disprove) this conjecture!*

Most Common Conjecture: The cycle structure of g is the same as the cycle structure of hgh^{-1}; that is, conjugates have the same cycle structure.

Exercise 5. Returning to the Rubik's Cube…

Recall that the generators of the group of configurations of the Rubik's Cube were described as:

F:=(17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11);
R:=(25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24);
U:=(1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19);
B:=(33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27);
D:=(41,43,48,46)(42,45,47,44)(22,30,38,14)(23,31,39,15)(24,32,40,16);
L:=(9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35);

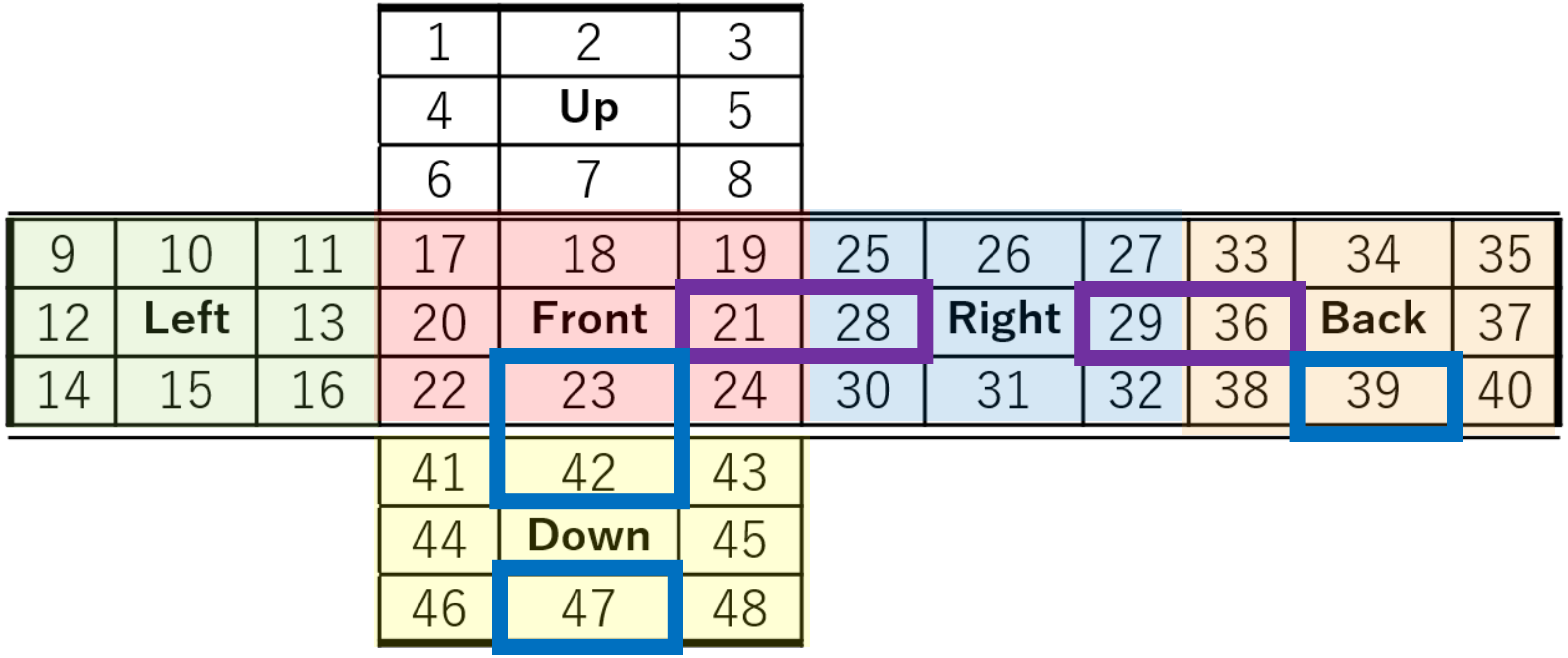
Find the permutation h in S_{48} having the property that $U = h*F*h^{-1}$. That is, prove that U and F are conjugate. Is h in the Rubik's Cube group?

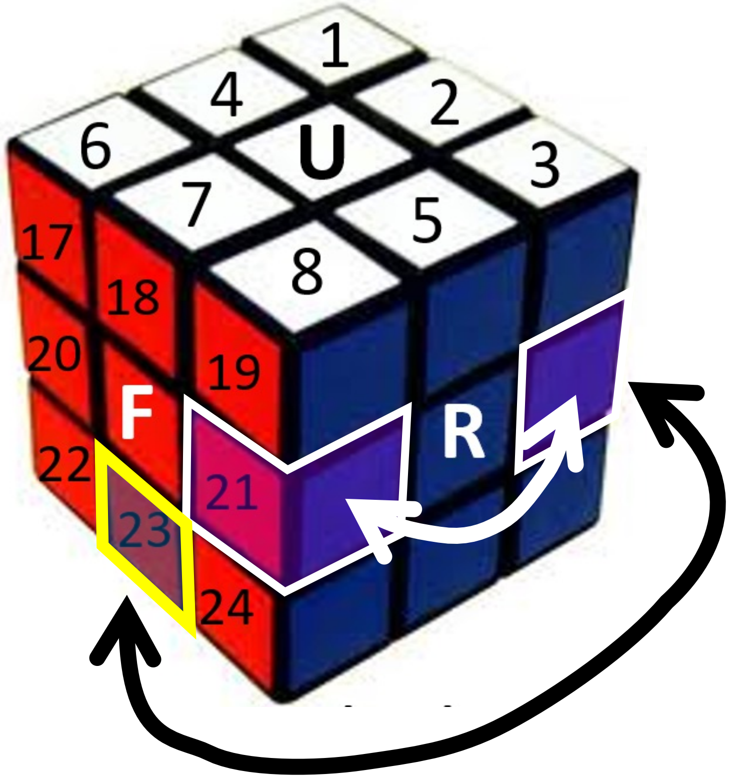
$$\left.\begin{aligned} h = &(1, 17, 41, 40)(2, 18, 42, 39)(3, 19, 43, 38)(4, 20, 44, 37) \\ &(5, 21, 45, 36)(6, 22, 46, 35)(7, 23, 47, 34)(8, 24, 48, 33) \\ &(9, 11, 16, 14)(10, 13, 15, 12)(27, 25, 30, 32)(26, 28, 31, 29) \end{aligned}\right\}$$

This moves the Up face to the Front face

Exercise 6.

Provide a geometric description of all elements in the conjugacy class of $(R^2D^2)^3 = (21, 36)(23, 39)(28, 29)(42, 47)$ in the Rubik's Cube group.





Students are encouraged to consider the difference between the conjugacy class of $(R^2D^2)^3$ in the Rubik's cube group versus the conjugacy class in S_{48}.

Goals of Project 3:

- To introduce students to the notion of *conjugation* and *conjugacy classes*

- To build the student's understanding of conjugation and cycle structure in S_n

- To give students an opportunity and the means to discover patterns, build intuition, and make conjectures.

How do the Projects fit into my Course?

I am a proponent of *active discovery learning* environments.

- Professors should be facilitators of knowledge and understanding; they are there to guide and stimulate the students, and to create environments in which students discover new ideas on their own.
- Learning is more meaningful if the student is allowed to experiment on their own rather than listening to the professor lecture every lesson.
- My course involves lecture, group work, student presentations, and projects. There are weekly written assignments.

How Class Time is Spent in Abstract Algebra

~ **30% -** Students (individually) presenting proofs or solutions to assigned theorems or exercises that were solved outside of class

~ **28% -** Class activities:

- Quiz based on reading
- Guided problem-solving
- Students (often in groups) presenting proofs or solutions to problems "on the spot"

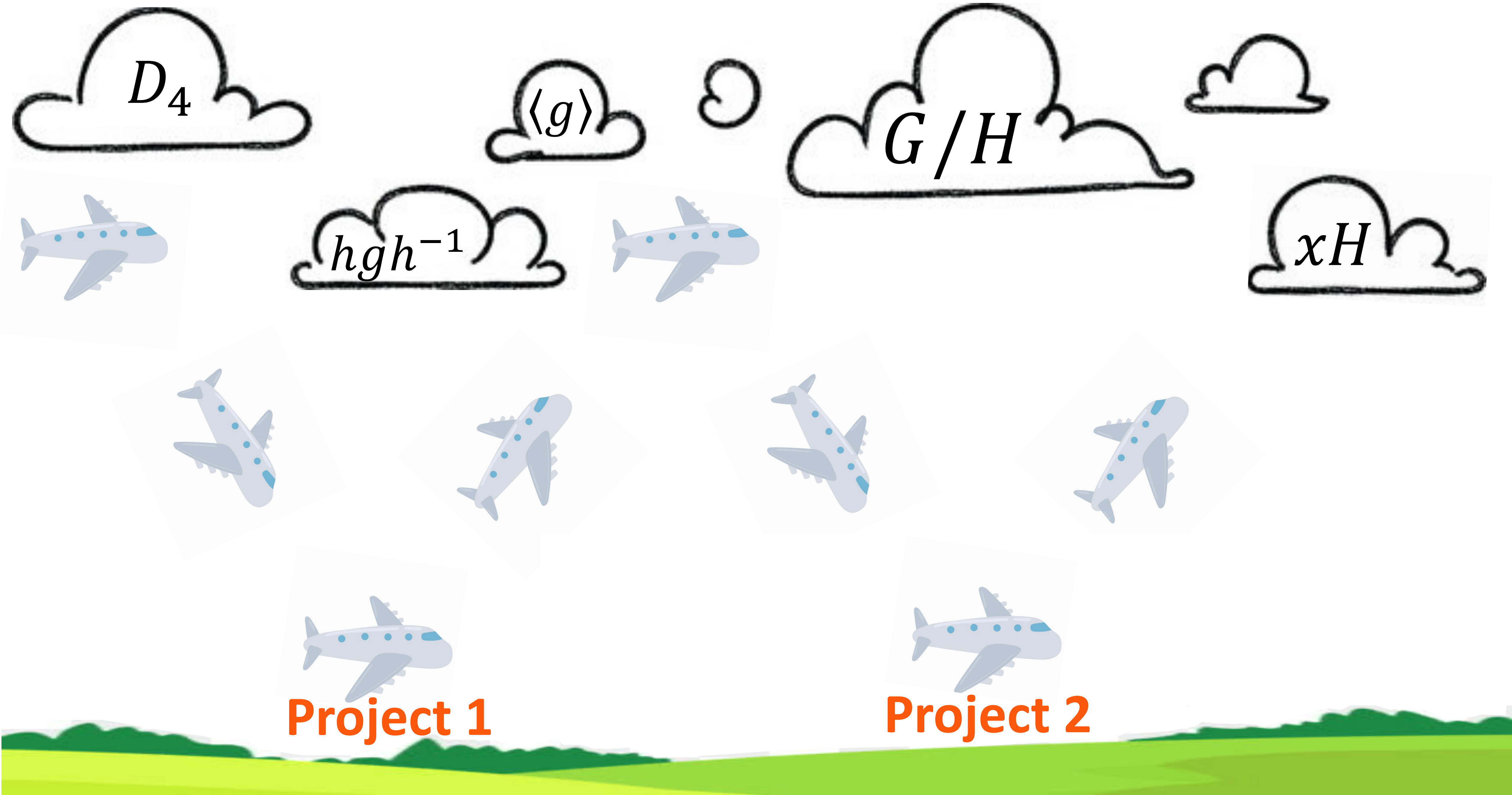
~ **25% -** Lecture (often driven by questions posed to class)

~ **12% - Projects (about 4-5 per semester)**

~ **5%** **-** Hourly exams

My General Philosophy underlying these Projects

- They are designed to provide intuition and inspire experimentation and play
- Projects are not meant to be difficult; most are (nearly) completed in class. My focus in the course is on proof-writing, which students find challenging.
- Since the projects reinforce material already covered, they provide time and space for students who are having difficulty with concepts and proof-writing
- The break from abstraction can boost the spirits of students who are feeling like “an idiot”


D_4
$\langle g\rangle$
G/H
hgh^{-1}
xH
Project 1
Project 2

Exercise: Imagine the strongest student(s) you have ever taught in Abstract Algebra. Describe their background.

Student 1:
Mother was a professional who worked A LOT; the student, an only child, taught himself to program at an early age and kept himself busy living in a computer-generated world. He now has a PhD in math.

Student 2:
Similar story, but not an only child…the student taught himself to program at an early age and spent many hours playing in a computer-generated world.

These students had been living in the world of abstraction for years; they really don't need the projects. (Including an open-ended question or two at the end of the project will keep these folks engaged.)

A Few Words about Projects in Abstract Algebra II

Aimed at a different audience, my Abstract II projects differ in nature from the Abstract I projects…

- Lengthier writing assignments whereby students prove a sequence of given statements to reach a desired result.
- Computational assignments meant to teach students computational content/methods (e.g., using Buchberger’s Algorithm to compute Groebner bases)
- Largely completed outside of class

References:

- E. Dubinsky, J. Dautermann, U. Leron, R. Zazkis. On Learning Fundamental Concepts of Group Theory. *Educational Studies in Mathematics* **27**, 267-305 (1994)
- G. Pellegrino. Sul Massimo ordine delle calotte in $S_{4,3}$. *Matematiche (Catania)* **25** 149-157 (1971)
- J. Holdener. "Product-free Sets in the Card Game Set," *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, **15**(4), 289-297 (2005)
- J. Lewis, An Analysis of Students' Difficulties in Learning Group Theory. Masters Thesis, Concordia University (2013)

Thank you!

Note: There is Better (Geometric) Framework for Analyzing SET…

Associate each card in the SET® deck with a point in the vector space

$$D = F_3 \times F_3 \times F_3 \times F_3$$

Fact: Three points $a, b, c \in F_3^4$ represent collinear points iff $a + b + c = 0$.

So 3 cards making up a SET correspond to lines in F_3^4.

The question becomes…

How big is the largest subset of F_3^4 containing no lines?

(This question was answered by Giuseppe Pellegrino in 1971.)