Using Conceptual Analysis as a Theoretical Tool to Resolve the Tension between Advanced and Secondary Mathematics: The Cases of Equivalence and Inverse

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Undergraduate Students' Reasoning about Equivalence in Multiple Mathematical Domains: Exploration and Theory Building

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Introduction

- **Advanced mathematics has long been considered an essential part of the mathematical preparation of secondary teachers.**
  - In 1932, Felix Klein argued that it was important for teachers to view "elementary mathematics from an advanced standpoint."
  - More recently, the CBMS (2012) reiterated this argument and outlined many specific ways in which this might occur:
    - "It would be quite useful for prospective teachers to see how C can be built as a quotient of R[x]" (p. 59).
- **"[A]dvanced mathematics serves to deepen, and more rigorously confirm, the specific mathematical ideas secondary teachers will teach" (Wasserman, 2018, p. 3).**

Introduction

- **While this idea seems imminently reasonable in theory, it has proven considerably difficult to implement in practice:**
  - “[A] school teacher’s knowledge of advanced mathematics, such as abstract algebra, [should] translate to their instructional practice in some way. And yet school mathematics teachers should not, in fact, end up teaching their students abstract algebra. This is a *difficult tension* to resolve” (Wasserman, 2017, p. 81, emphasis added).

Introduction

- **While this idea seems imminently reasonable in theory, it has proven considerably difficult to implement in practice:**
  - "The CBMS (2012) recommendations for the mathematical preparation of teachers [include] statements like, 'it would be quite useful for prospective teachers to see how **C** can be built as a quotient of **R**[x]' (CBMS, 2012, p. 59). A very reasonable question to ask in response to such a statement is 'Why?' Abstract algebra certainly provides a highly sophisticated perspective on a variety of secondary mathematics topics, but it simply does not follow that a teacher's pedagogical practice would (or even could) benefit from studying abstract algebra. Or perhaps, rather, we should say it does not follow simply." (Larsen et al, 2018, p. 74)

Introduction

- **We take a pragmatic stance in this debate:**
  - "Basic knowledge of group theory is in fact neither necessary nor obligatory for addressing the (more elementary) mathematics. Nevertheless, […] it can be helpful" (Zazkis & Marmur, 2018, p. 379).
  - "Any exposure to or instruction about, say, abstract algebra content, is not innately relevant for secondary teachers […]. But that is not the same as claiming that it cannot be beneficial" (Wasserman, 2018, p. 6).
  - Since prospective teachers will continue to be required to take advanced mathematics courses, let's focus on how to make them ***as useful as possible***.

Introduction

- **In this talk, we are going to illustrate one way we've found for making advanced mathematics as useful as possible for prospective teachers.**
  - The name of this approach is ***conceptual analysis***, which we'll talk about here in a minute.

Consider the following abstract algebra tasks

Task 1.1: Is $\phi\colon \mathbb{Q} \to \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$ a function? Explain.

Task 1.2: Is $g\colon \mathbb{Q} \to \mathbb{Q}$ given by $g\left(\frac{a}{b}\right) = \frac{a+b}{b}$ a function? Explain.

Task 1.3: Is $f\colon \mathbb{Z}_4 \to \mathbb{Z}$ given by $f([a]_4) = a$ a function? Explain.

Consider the following abstract algebra tasks

Task 2.1: Prove: for all $a, b \in \mathbb{Z}_3[i]$, all equations of the form $a + x = b$ have a unique solution in $\mathbb{Z}_3[i]$.

Task 2.2: Prove: for all $a \in \mathbb{Z}_3[i] \backslash \{0\}$ and $b \in \mathbb{Z}_3[i]$, all equations of the form $ax = b$ have a unique solution in $\mathbb{Z}_3[i]$.

The Main Point:

- **These tasks, despite having little obvious relevance for secondary mathematics, can indeed be a useful opportunity for prospective teachers to make connections to secondary mathematics.**
  - "A very reasonable question to ask in response to such a statement is 'Why?'" (Larsen et al., 2018, p. 74).
  - Our answer: because the underlying ***ways of reasoning*** needed to complete these tasks mirror ways of reasoning that are productive in secondary mathematics.

Conceptual Analysis

- **A conceptual analysis is an explicit description of the ways in which someone reasons about a particular mathematical idea (Thompson, 2008).**
  - **How is conceptual analysis useful here?**
    - It focuses our attention on the underlying ways of reasoning students need to complete tasks (instead of on surface-level differences in content)

Overview

- Present conceptual analyses for the key ideas of equivalence and inverse using examples from secondary mathematics
- Provide excerpts from a series of interviews we conducted with prospective teachers in which they use these ways of reasoning while working on the aforementioned abstract algebra tasks
- Conclude with practical takeaways for those who teach and design these courses

Conceptual analysis: equivalence

- **How might students productively interpret/reason about "A is equivalent to B"?**
  - ***Common characteristic***: A and B are equivalent because they share a key characteristic
    - The expressions 3(x+1)+1 and 3x+4 are equivalent because they have the same value
  - ***Transformational:*** A and B are equivalent because A can be transformed into B using a set of allowed rules/procedures
    - The expressions 3(x+1)+1 and 3x+4 are equivalent because distributing 3(x+1)+1 and then combining like terms yields 3x+4

Conceptual analysis: equivalence

- **How might students productively interpret/reason about "A is equivalent to B"?**
  - ***Common characteristic*:** A and B are equivalent because they share a key characteristic
    - The equations 3x+4=4 and 3x=0 are equivalent because they share the same solution set
  - ***Transformational:*** A and B are equivalent because A can be transformed into B using a set of allowed rules/procedures
    - The equations 3x+4=4 and 3x=0 are equivalent because subtracting 4 from both sides of 3x+4=4 yields 3x=0

Conceptual analysis: inverse

- **How might students productively interpret/reason about inverses?**
  - ***Inverse as an undoing***: an operation is applied to undo the effects of the previous operation
    - Multiplication by 4 can be 'undone' by dividing by 4
  - ***Inverse as a manipulated element:*** an inverse element is produced by applying a suitable procedure to the given element
    - The multiplicative inverse of 4 is obtained by taking the reciprocal, 1/4
  - ***Inverse as a coordination:*** attends to the fact that the combination of an element and its inverse with respect to the binary operation is the identity
    - 4 and ¼ are multiplicative inverses because 4*(1/4)=1

Conceptual analysis: inverse

- **How might students productively interpret/reason about inverses?**
  - ***Inverse as an undoing***: an operation is applied to undo the effects of the previous operation
    - For $f(x)=x+5$, since $f(3)=8$, then $f^{-1}(8)=3$.
  - ***Inverse as a manipulated element:*** an inverse element is produced by applying a suitable procedure to the given element
    - Switching and solving: $x=y+5$ <=> $y=x-5$, so $f^{-1}(x)=x-5$
  - ***Inverse as a coordination:*** attends to the fact that the combination of an element and its inverse with respect to the binary operation is the identity
    - $f(x)=x+5$ and $g(x)=x-5$ are inverse functions because $f(g(x))=x=g(f(x))$.

Conceptual analysis: inverse

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    - f(x)=x+5 and g(x)=x-5 are inverse functions because f(g(x))=x=g(f(x)).

Overview

- Present conceptual analyses for the key ideas of equivalence and inverse using examples from secondary mathematics
- **Provide excerpts from a series of interviews we conducted with prospective teachers in which they use these ways of reasoning while working on the aforementioned abstract algebra tasks**
  - ***Equivalence*: common characteristic, transformational**
  - ***Inverse*: undoing, manipulated element, coordination**
- Conclude with practical takeaways for those who teach and design these courses

UTILITY OF CONCEPTUAL ANALYSIS

Equivalence (Cook, Reed, & Lockwood, 2022)

Interpretation of equivalence	Description	Examples from school mathematics	
		Algebraic expressions	Algebraic equations
Common characteristic	Involves interpreting or determining the sameness of objects in terms of a feature that the objects share	Expressions "f and g are equivalent [because] $f(x) = g(x)$ for all x in the common domain" (Solares & Kieran, 2013, p. 122).	Equations are equivalent when they share the same solution set (e.g., Alibali et al., 2007).
Transformational	Involves interpreting or determining equivalence on the basis that one object can be manipulated into the other pursuant to an established procedure or set of actions, rules, or properties	Expressions are equivalent if one can be transformed into the other by using certain algebraic rules (e.g., Zwetzschler & Prediger, 2013).	Equations are considered equivalent when one can be manipulated into the other according to certain algebraic rules (e.g., Baiduri, 2015).

UTILITY OF CONCEPTUAL ANALYSIS

Equivalence

Task 1.1: Is $\phi: \mathbb{Q} \to \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$ a function? Explain.

No, because, for example:

$$\phi\left(\frac{1}{2}\right) = 1 + 2 = 3$$

$$\phi\left(\frac{2}{4}\right) = 2 + 4 = 6$$

UTILITY OF CONCEPTUAL ANALYSIS

Equivalence

Task 1.2: Is $g: \mathbb{Q} \to \mathbb{Q}$ given by $g\left(\frac{a}{b}\right) = \frac{a+b}{b}$ a function? Explain.

Yes, because:

$$g\left(\frac{a}{b}\right) = \frac{a+b}{b}$$

$$= \frac{a}{b} + \frac{b}{b}$$

$$= \frac{a}{b} + 1$$

UTILITY OF CONCEPTUAL ANALYSIS

Equivalence

Task 1.3: Is $f: \mathbb{Z}_4 \to \mathbb{Z}$ given by $f([a]_4) = a$ a function? Explain.

No, because, for example:

$$f([1]_4) = 1$$

$$f([5]_4) = 5$$

EQUIVALENCE: COMMON CHARACTERISTIC

Task 1.1: Is $\phi: \mathbb{Q} \to \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$ a function?

$\phi : \mathbb{Q} \to \mathbb{Z}$

$\phi\left(\frac{a}{b}\right) = a + b$

$\phi\left(\frac{2}{3}\right) = 2 + 3 = 5$

$$\phi\left(\frac{4}{6}\right) = 4 + 6 = 10$$

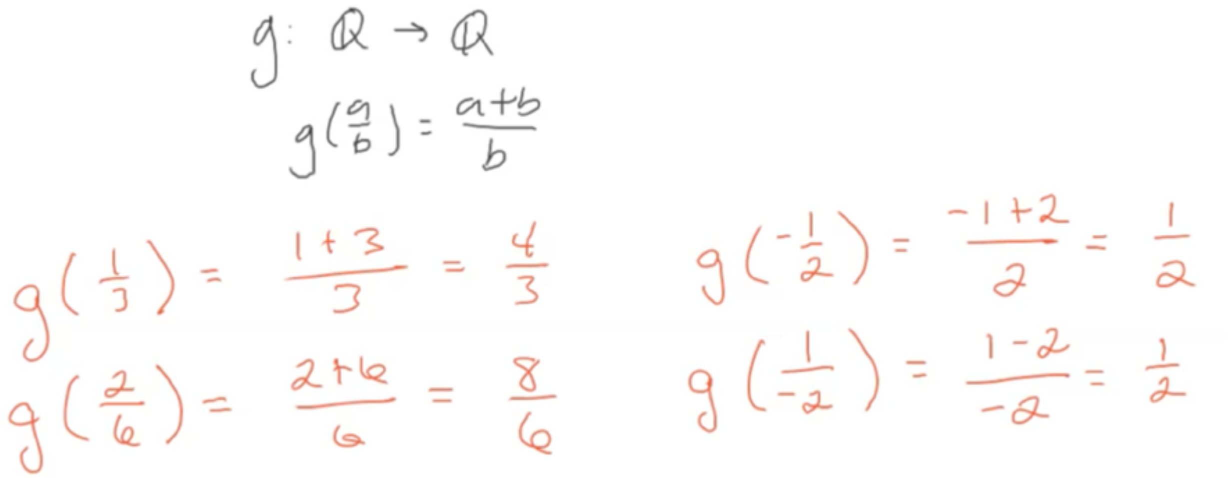
"Two over three, well, that's the same thing as four over six, but they would map to a different element. […] And so now you no longer have a function […] because […] 2/3 and 4/6 are essentially **equivalent**."

"I think of equivalence as, as I am comparing this object […] with this object […] and I'm just seeing if there's one property that they **share in common**. And if they have that property in common, then I would say that they're equivalent."

"they might not look the same but they have the **same property**"

EQUIVALENCE: COMMON CHARACTERISTIC

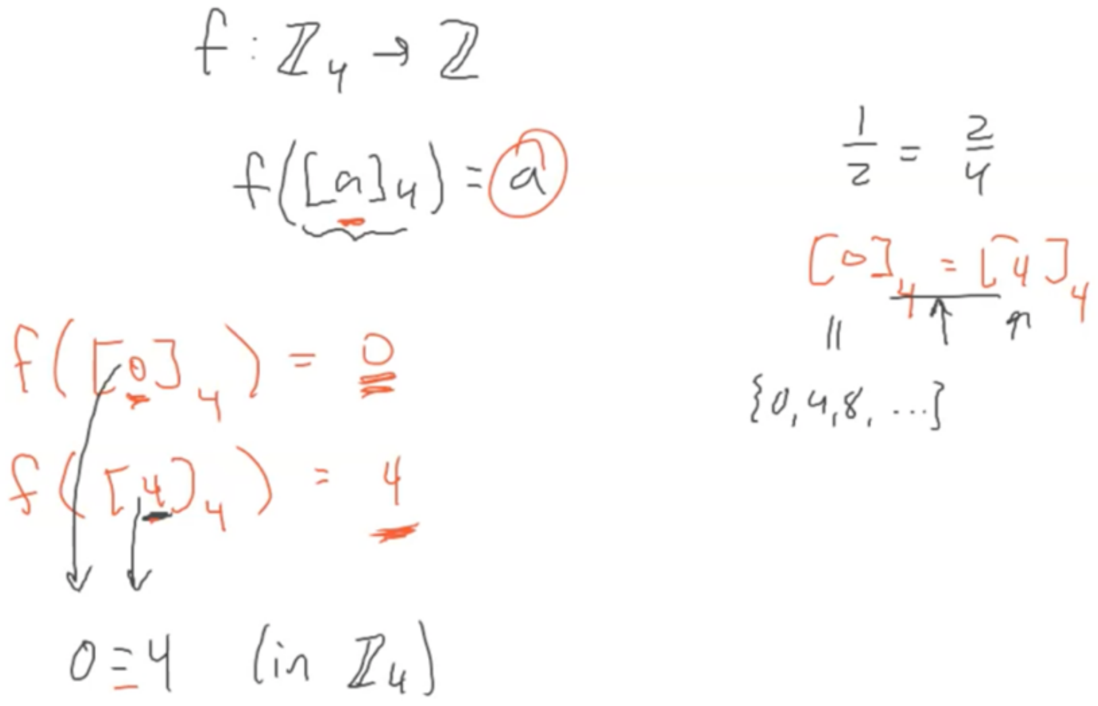
Task 1.2: Is $g: \mathbb{Q} \to \mathbb{Q}$ given by $g\left(\frac{a}{b}\right) = \frac{a+b}{b}$ a function?



"I would maybe see if, like, some sort of element would map to two different elements, right? And in this case, I would probably pick an element that is 'equivalent.' So I could probably pick like 1/3 or something. I guess I would maybe do it like $g(1/3)$ […] which is 4/3. And then see if I can find something that's like kind of equivalent to 1/3. So maybe like $g(2/6)$ […] which would be 8/6. […] 4/3 is the same thing as 8/6."

EQUIVALENCE: COMMON CHARACTERISTIC

Task 1.3: Is $f : \mathbb{Z}_4 \to \mathbb{Z}$ given by $f([a]_4) = a$ a function?



"You have 0, 4, 8 or whatever. They're all the exact same thing in $\mathbb{Z}_4$, but in my outputs I'm getting different values […] So that would be my reasoning to say like, oh, like, bam, no, not a function"

"With four, then they're all the same thing. So like zero, and four, and eight, you know, **they're all evenly divided by four**."

"Kind of like the Example 1 […] how I was, you know, **picking fractions that ultimately looked different but represented the same thing**, or the same property or those sameness, or whatever. **I wanted to do the exact same thing in this case**. You know, 0 and 4, obviously, in Z mod 4, they're both zero. They're the exact same thing even though they look different."

EQUIVALENCE: TRANSFORMATIONAL

Task 1.1 Is $\phi: \mathbb{Q} \to \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$ a function?

$\phi: \mathbb{Q} \to \mathbb{Z}$

$\phi\left(\frac{a}{b}\right) = a + b$

$\phi\left(\frac{2}{3}\right) = 2 + 3 = 5$

$$\frac{4}{6} = \frac{2}{2}\left(\frac{2}{3}\right) = \frac{2}{3}$$

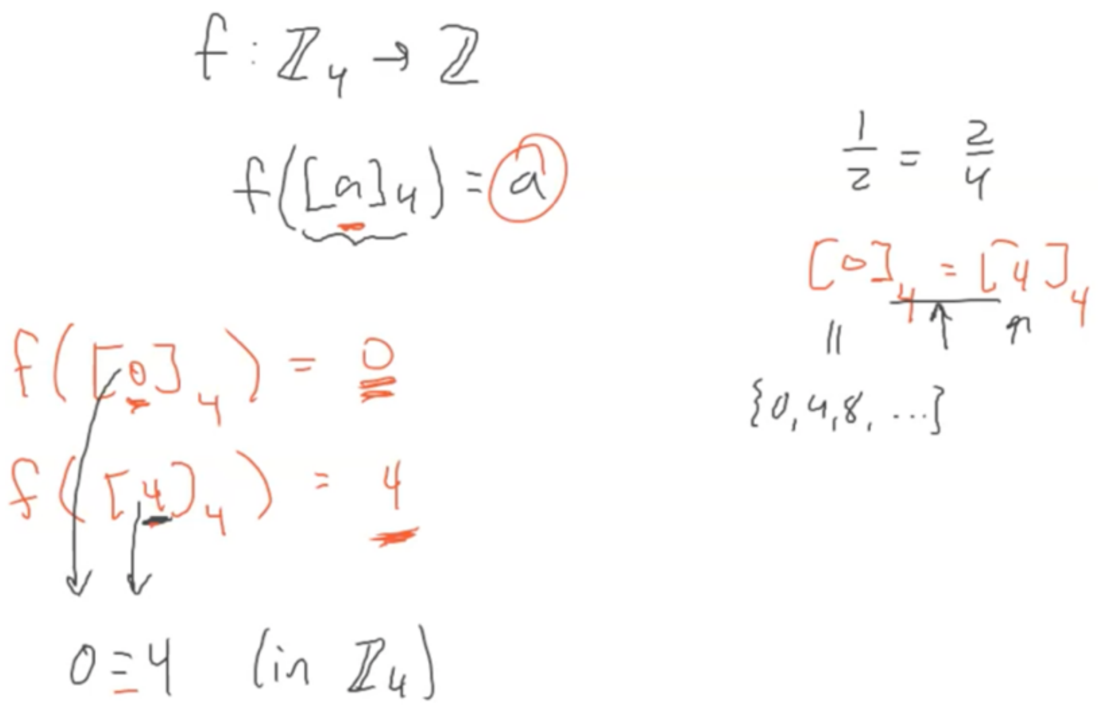
"I could **reduce one** or think of them [the fractions 2/3 and 4/6] as the same thing"

To identify if elements are equivalent: "take a fraction and you see if you can **simplify** it all the way down"

"ultimately I could reduce one or think of them as the same thing."

EQUIVALENCE: TRANSFORMATIONAL

Task 1.3: Is $f : \mathbb{Z}_4 \to \mathbb{Z}$ given by $f([a]_4) = a$ a function?



“0 in $\mathbb{Z}_4$ is the same thing [as] 4 in $\mathbb{Z}_4$, right? […] You know, I’m just, it’s kind of like those fractions. We kind of **reduce** them down [using repeated addition/subtraction of the modulus 4].”

UTILITY OF CONCEPTUAL ANALYSIS

Equivalence

- Ways of reasoning that support productive engagement with such tasks in abstract algebra mirror (and can thus reinforce) the same ways of reasoning needed to reason productively about equivalence in secondary mathematics.
  - Common characteristic
    - $\mathbb{Q}$: shared value of quotients when dividing numerator by denominator
    - $\mathbb{Z}_4$: shared remainders after division by 4
  - Transformational
    - $\mathbb{Q}$: fraction reduction, i.e., division of numerator and denominator by their GCD
    - $\mathbb{Z}_4$: repeated addition/subtraction of modulus 4

UTILITY OF CONCEPTUAL ANALYSIS

Inverse (Cook et al., 2022)

Way of reasoning	Description	Examples from school mathematics	
		Multiplicative inverses in $\mathbb{R}$	Compositional inverses of functions
Inverse as an undoing	"inverse is associated with an operation that cancels the previous operation and 'returns to the starting point'" (Zazkis & Kontorovich, 2016, p. 107)	"An inverse is something that will return you to the starting point. Let's say I pushed the wrong button on the calculator and multiplied by 5. For correcting this, I need to divide by 5" (Kontorovich & Zazkis, 2017 p. 31).	An inverse function is "the operation needed to go in the reverse direction, from the final state to the initial state" (Vergnaud, 2012, p. 441).
Inverse as a manipulated element	Viewing inverse in terms of a procedure by which an element is changed into its inverse element.	The multiplicative inverse of any nonzero real number can be found by taking its reciprocal (e.g., Clay et al., 2012).	An inverse function can also be viewed in terms of "switching the x and y variables and solving for y" (Pinto & Schubring, 2018, p. 900).
Inverse as a coordination	Conceiving of inverse as a relationship between two elements such that the combination of those two elements via the relevant binary operation yields the identity element.	"We remember multiplication if we take a number and multiply it by its multiplicative inverse you will get the multiplicative identity 1." (Clay et al., 2012, p. 769).	The composition of a function with its inverse function yields the identity function (e.g., Vidakovic, 1996).

UTILITY OF CONCEPTUAL ANALYSIS

Inverse (Uscanga & Cook, 2017)

PSTs (Josh and Meagan) explored the algebraic structure of $\mathbb{Z}_3[i]$ (the finite field of order 9).

Task 2.1: Prove: for all $a, b \in \mathbb{Z}_3[i]$, all equations of the form $a + x = b$ have a unique solution in $\mathbb{Z}_3[i]$.

Task 2.2: Prove: for all $a \in \mathbb{Z}_3[i]\backslash\{0\}$ and $b \in \mathbb{Z}_3[i]$, all equations of the form $ax = b$ have a unique solution in $\mathbb{Z}_3[i]$.

INVERSE: UNDOING

Task 2.1: Prove all equations of the form $x + a = b$ have a unique solution in $\mathbb{Z}_3[i]$.

- When attempting to identify a solution candidate (the ‘existence’ part of the proof), Josh transformed $x + (c + di) = (a + bi)$ into $x = (a - c) + (bi - di)$.
  - Josh: “you add the inverse of $[c + di\ ]$”
  - *Inverse as an undoing*: using algebraic manipulation to *undo* an operation

Meagan:	**We would be able to eliminate, um, the $c + di$.**
Researcher:	Mmhmm.
Meagan:	We would be able to get x by itself, so then we could solve.
Researcher:	What do you mean by eliminate?
Meagan:	Um, getting rid of, or, like, um, oh gosh.
Josh:	Like **cancelling it out**.
Meagan:	**Cancelling it out**. So then you could get x by itself.

INVERSE: MANIPULATED ELEMENT

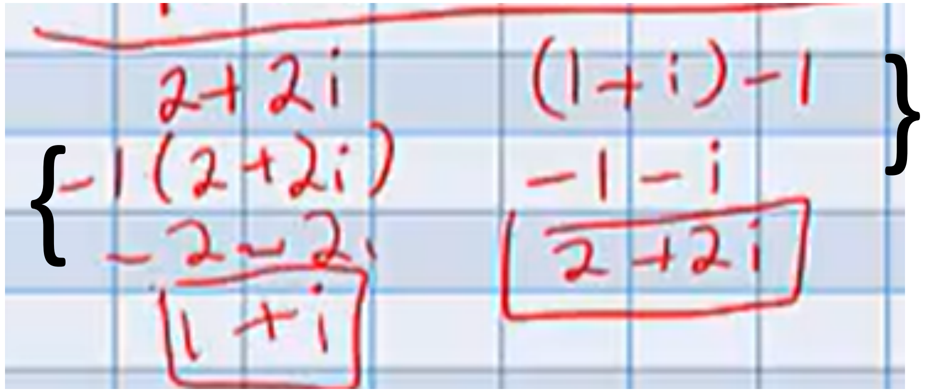
Task 2.1: Prove all equations of the form $x + a = b$ have a unique solution in $\mathbb{Z}_3[i]$.

- When asked how they could be sure that such an element $-c - di$ existed for each element $c + di$ in $\mathbb{Z}_3[i]$, Josh responded that "you multiply it by negative one" and then "simplify it from there", i.e.:
  1) Start with an element
  2) Multiply it by -1
  3) Use congruence mod 3
  4) The result is the inverse element

Inverse as a manipulated element: viewing the inverse relationships in terms of inverse *elements* that were obtained by manipulating the original element via a procedure

Multiply by -1

Use congruence mod 3



Multiply by -1

Use congruence mod 3

INVERSE: COORDINATION

Task 2.2: Prove: for all $a \in \mathbb{Z}_3[i] \setminus \{0\}$ and $b \in \mathbb{Z}_3[i]$, all equations of the form $ax = b$ have a unique solution in $\mathbb{Z}_3[i]$.

·	0	1	2	i	2i	1+i	1+2i	2+i	2+2i
0	0	0	0	0	0	0	0	0	0
1	0	1	2	i	2i	1+i	1+2i	2+i	2+2i
2	0	2	1	2i	i	2+2i	2+i	1+2i	1+i
i	0	i	2i	2	1	2+i	1+i	2+2i	1+2i
2i	0	2i	i	1	2	1+2i	2+2i	1+i	2+i
1+i	0	1+i	2+2i	2+i	1+2i	2i	2	1	i
1+2i	0	1+2i	2+i	1+i	2+2i	2	i	2i	1
2+i	0	2+i	1+2i	2+2i	1+i	1	2i	2+i	2
2+2i	0	2+2i	1+i	1+2i	2+i	i	1	2	2i

Meagan's multiplication table for $\mathbb{Z}_3[i]$ in which she circles pairs of elements that multiply to produce 1

Meagan: Oh! Okay!

Josh: Ok, yeah. It's all you.

Meagan: Oh my gosh! Because we just said anything times itself should equal 1, right? So we have all the ones that equal 1. One, one, one, one. [**She begins circling all the 1's that appear as entries in the multiplication table**].

Researcher: You said that anything times itself should equal 1?

Meagan: Ok, not times itself. Lightbulb! Ok, so [...] **any number times its inverse should equal 1**.

INVERSE: COORDINATION

Task 2.2: Prove: for all $a \in \mathbb{Z}_3[i] \backslash \{0\}$ and $b \in \mathbb{Z}_3[i]$, all equations of the form $ax = b$ have a unique solution in $\mathbb{Z}_3[i]$.

·	0	1	2	i	2i	1+i	1+2i	2+i	2+2i
0	0	0	0	0	0	0	0	0	0
1	0	1	2	i	2i	1+i	1+2i	2+i	2+2i
2	0	2	1	2i	i	2+2i	2+i	1+2i	1+i
i	0	i	2i	2	1	2+i	1+i	2+2i	1+2i
2i	0	2i	i	1	2	1+2i	2+2i	1+i	2+i
1+i	0	1+i	2+2i	2+i	1+2i	2i	2	1	i
1+2i	0	1+2i	2+i	1+i	2+2i	2	i	2i	1
2+i	0	2+i	1+2i	2+2i	1+i	1	2i	2+i	2
2+2i	0	2+2i	1+i	1+2i	2+i	i	1	2	2i

Meagan's multiplication table for $\mathbb{Z}_3[i]$ in which she circles pairs of elements that multiply to produce 1

Inverse as a coordination

Josh: "So basically everywhere that we get a 1, because of our table, those are going to be the inverse pairs."

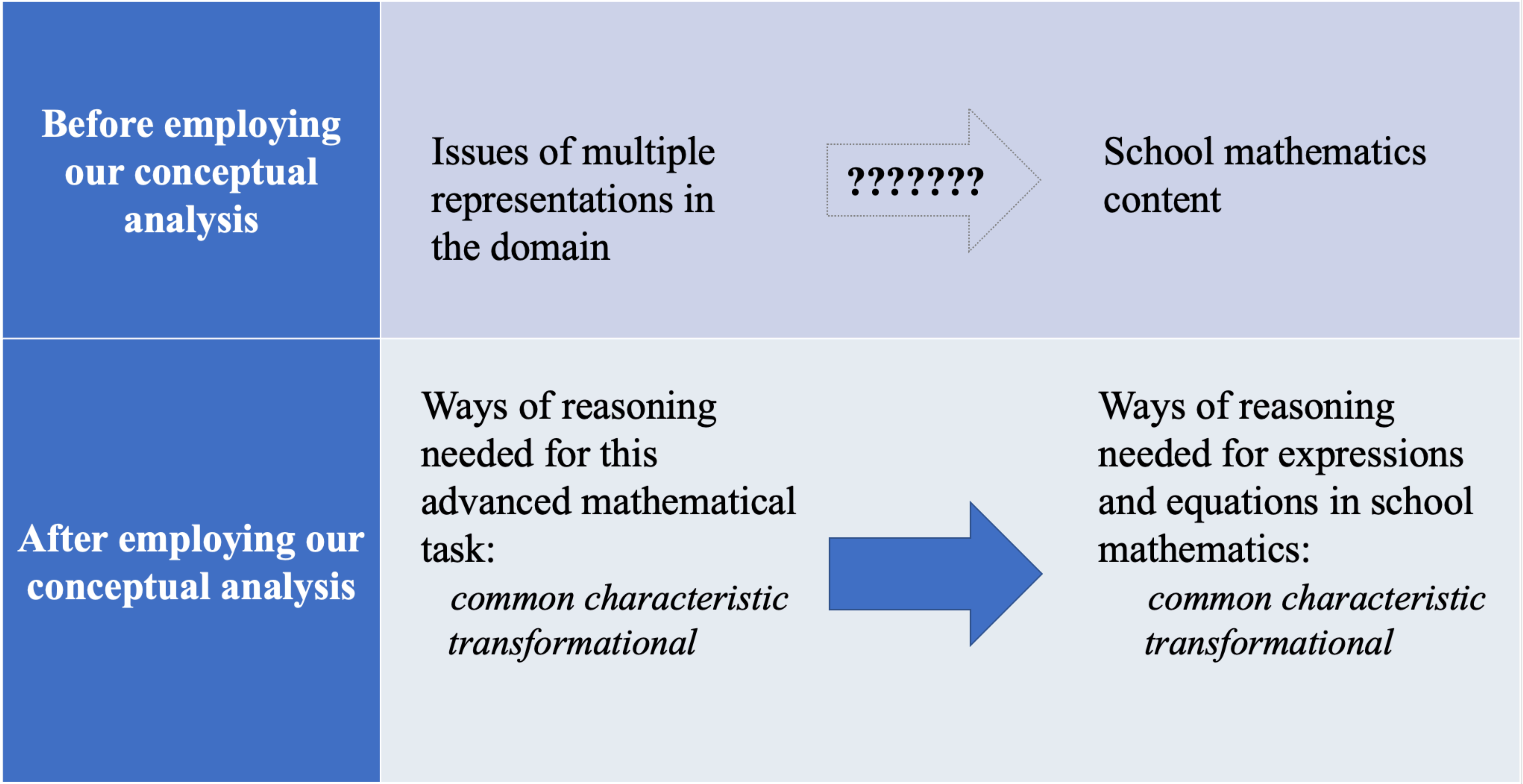
Binary operation: focus on *pairs* of elements that multiply to yield the multiplicative identity, 1

Identity: recognizing that multiplying inverse elements yield the identity, 1

Set: recognizing that an element and its inverse are both in $\mathbb{Z}_3[i]$

Discussion

- **Though advanced mathematics might not be inherently useful for teaching secondary mathematics, we should work to make advanced mathematics courses *as useful as possible***
- **Our approach: conceptual analysis**
  - *Equivalence*: common characteristic, transformational
  - *Inverse*: undoing, manipulated element, coordination
- **The ways of reasoning about equivalence and inverses that were useful in completing tasks in advanced mathematics mirror those needed reason productively in secondary mathematics**


Before employing our conceptual analysis
Issues of multiple representations in the domain
???????
School mathematics content
After employing our conceptual analysis
Ways of reasoning needed for this advanced mathematical task:
common characteristic
transformational
Ways of reasoning needed for expressions and equations in school mathematics:
common characteristic
transformational

Before employing our conceptual analysis	Investigating and proving conjectures about the the finite field of order 9	???????	School mathematics content
After employing our conceptual analysis	Ways of reasoning needed for this advanced mathematical task: *inverse as an undoing* *inverse as a manipulation* *inverse as a coordination*		Ways of reasoning needed for inverses in school mathematics: *inverse as an undoing* *inverse as a manipulation* *inverse as a coordination*

Discussion

- **Key takeaway: rather than focusing on surface-level differences in content, let's focus instead on the underlying ways of reasoning**
  - Lots of resources in the mathematics education literature about productive ways of reasoning for particular topics

- **Future work: here we have outlined our image of potential connections prospective teachers might be able to make by attending to similarities in their ways of reasoning. But we haven't yet studied how this might work and what kinds of connections and commonalities prospective teachers themselves might actually attend to.**

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