Establishing Sustainable Active Learning in Linear Algebra

Marie MacDonald

March 21 - OLSUME

Active Learning at Cornell

- Cornell University is encouraging Active Learning Initiatives (ALI) and better pedagogical approaches.
- The Center for Teaching Innovation has grants for ALI that are funded through the university and a private donation.
- The Math department has held two ALI grants.
- Cornell just celebrated 10 years of ALI.

Active Learning in Math

			
Tara Holm	Steve Bennoun	Tim Riley	Marie MacDonald
PI	ALI Lecturer	PI	ALI Lecturer
2017-2020	2017-2020	2020-2023	2020-2022

Many more faculty and graduate students have contributed to ALI.

Active Learning Lecturer in Math

My two main goals as ALI lecturer were:

- Increase student engagement during class time.
- Add some assessment that is more representative of Math in 2023.

Active Learning in Math

- There are at least 1400 students enrolled in Math ALI courses yearly.
- We have created a bank of materials with instructions to teach with AL.
- Instructors can choose how much they want to use, our broad goal is to increase student engagement during class time.
- This talk will focus on Linear Algebra (LA), but we have materials for a wide range of courses.

Linear Algebra at Cornell

- We have redesigned two Linear Algebra courses MATH 2210 and MATH 4310.
- Our LA course with largest enrollment is MATH2940 Linear Algebra for Engineers, it is trickier to change that course.
- There are a few more specialized LA courses.
- Enrollment has been up in all LA courses.

MATH 2210

- Often the first math course math majors will take.
- Introductory LA with computations and light proofs.
- 4 sections in the fall and 2 in the spring, about 60 students per section.
- Common assessment for all sections.

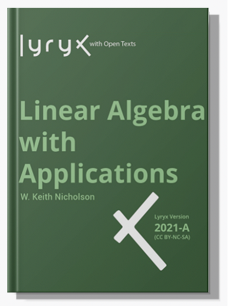
MATH 4310

- Mostly third and fourth year students.
- Students are advised to take the honors version if they want to go to grad school in math.
- Econ, CS and math with CS concentration majors.
- Proof based LA over general fields.
- 45 students in the fall, 60 in the spring with 2 sections.
- In the past the course had only been taught with lectures.

MATH 2210 - ALI

- Switched to a free open source textbook.
- Created a companion notebook for instructors and students that includes polling questions and in-class group activities.
- Application workshops in discussion and projects.
- Programming demonstrations using MATLAB.
- The course has gone through 6 semesters under the restructure.
- The new materials have been successful both online (synchronous) and in person.

MATH 2210 - Textbook



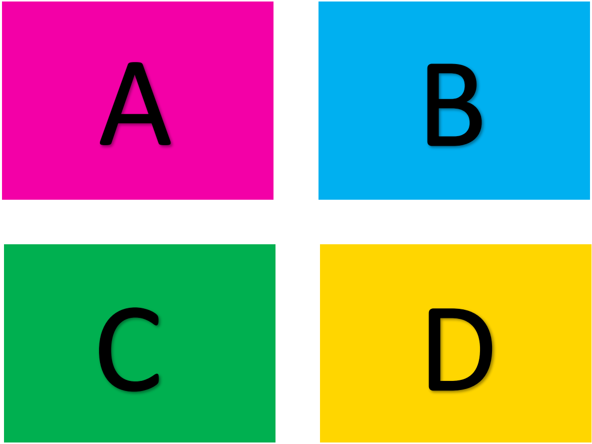
- *Linear Algebra with Applications* by W. Keith Nicholson.
- Accessible through the Lyryx website, free to share and adapt.
- Was formally a published textbook, has good exercises.

MATH 2210 - Notebook

- Notes that align with the textbook but with different examples.
- They contain partial definitions and theorem.
- Examples and proofs are left blank for in-time completion.
- The notes contain polling questions and group activities.

MATH 2210 - Polling

For in person classes we have been using low-technology polling.



Polling Example - 1

How do you feel about using Active Learning in your courses?

A 

B 

C 

D 

A B
C D


Notebook - Blanks

The blanks, make instructors slow down.

Definition 3.4. *If A is an $n \times n$ matrix, a number λ is called an* .... *of A if*

$$A\boldsymbol{x} = \lambda \boldsymbol{x} \text{ for some vector } \boldsymbol{x} \neq \mathbf{0} \text{ in } \mathbb{R}^n.$$

In this case, $\boldsymbol{x}$ is called an .... *of A corresponding to the eigenvalue λ, or a λ-eigenvector for short.*

Question 3.3.A. Which of the following is an eigenvector of the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$?

A $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

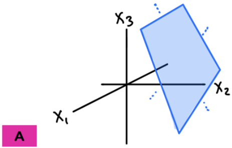
C $\begin{bmatrix} -3 \\ -3 \end{bmatrix}$

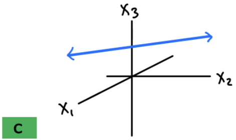
B $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$

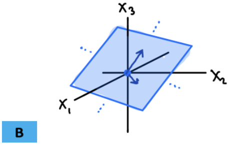
D $\begin{bmatrix} 1 \\ -2/3 \end{bmatrix}$

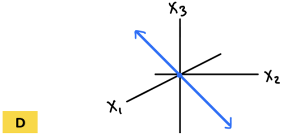
Polling Example - 2

Question 1.3.L. Which of the following could be the solution set to the linear system $[1 \ \ -1 \ \ 2 \mid 0\,]$?









Notebook - Group Work 1

Example 4.2.R. Find the shortest distance from the point $P = (3, 2, 3)$ to the plane $2x + y + 2z = 2$, and find the point Q on the plane that is the closest to P.

In your group:

- Draw a sketch of the situation (note that P is not on the plane).
- Hint: you will need to find a projection vector this will help you find Q.
- Hint: to find a point on the plane you can set 2 of the 3 coordinates to be zero, for example $y = z = 0$ and determine x.
- Once you know Q calculate the length of the vector $\overrightarrow{PQ}$.

Notebook - Group Work 2

Question 3.3.S. Complete the following table with four examples:

	diagonalizable	not diagonalizable
A^{-1} exists		
A^{-1} does not exist		

Notebook - Group Work 3

Example 1.2.M. Students were given the following problem:

Let A and B be $n \times n$ matrices and C be the $n \times 2n$ matrix $C = \begin{bmatrix} A & B \end{bmatrix}$. If $\text{rank}(A) = n$, prove that $\text{rank}(C) = n$ or show a counterexample.

In your group:

- Go over the three solutions below and grade them out of 5 points.
- Clearly explain why you took off or gave points.

Solution 1

Let A be the 2×2 matrix $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and B be the 2×2 matrix $\begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$. C must therefore $= \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 0 & b & a \end{bmatrix}$. Rank A is equivalent to the number of leading ones of A (which we can see to be 1 by inspection). However, when we look at C, we find 2 leading ones (in columns 1 and 3). Thus $\text{rank}(C) \neq n$ in general.

Workshops

- Students have weekly discussion sections (tutorials).
- We recommend using 5 of these for workshops.
- There is a bank of guided worksheets on various topics.
- Some instructors have students submit the workshops and grade them (on completion).
- Some write follow up homework problems.
- Workshop material is fair game on test.

Topics

- 1.2 Polynomial Interpolation
- 1.3 Balancing Chemical Reactions
- 2.2 Markov Chains
- 3.1 Polygon Area
- 3.2 Lights Out
- 3.3 Complex Eigenvalues
- 3.3 Graphs
- 3.3 Linear Recurrences
- 5.4 Hamming Codes
- 5.6 Curve Fitting
- 6.2 Abstract Algebra
- 6.2 Group Representations
- 6.3 Enchanted Squares
- 6.4 Integration

Curve Fitting

Math 2210 Workshop

Curve Fitting

Real-life data can be messy and does not necessarily fit a nice or simple function. In these cases, if we wish t model our data, we will need to approximate it by a chosen function, minimising the error between our data and our model (here, in a least-squares sense). We can use linear algebra to solve these best-approximation problems and make predictions in important applications, such as meteorology.

Warmup

Question 1. Compute the line of best fit for the following points by using a least squares approximation:

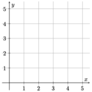
$$(1,0),(1,2),(2,3),(3,1),(5,4)$$

(a) We wish to find a line $y = r_0 + r_1 x$ which lies as close as possible to these points. To do this, we solve the normal equation:

$$\mathbf{A}^{\intercal}\mathbf{A}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \mathbf{A}^{\intercal}\mathbf{y}$$

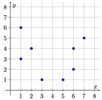
What should we take as our matrix $\mathbf{A}$? (**Hint:** The normal equations give us the best approximation t a solution to $\mathbf{A}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \mathbf{y}$, where each row of this system corresponds to one of our points.)

(b) Solve the normal equations to determine the equation of the best fit line. Graph this line, along with al of the points below.



Fitting Other Curves

In some cases, finding a best fit line provides us with a good understanding of what we are modeling, as there is an underlying linear relationship between the variables. However, this is not always the case. Consider the following data:



In this case, our data does not very close to any line. However, the data is higher for the small and large values of x and smaller for moderate values of x, so may be fit well by a parabola.

Question 2.

(a) We'd like to determine the coefficients of the parabola equation

$$y = r_0 + r_1 x + r_2 x^2$$

which best represents our data (in a least-squares sense). Present this in the form of a matrix equation $\mathbf{A}\mathbf{r} = \mathbf{y}$ for the data points given on the above axes.

(b) Use the normal equations to recast this best-approximation problem as solving a linear system. Solve this system to determine the coefficients $\mathbf{r}$ for the best-fit parabola. Round your coefficients to 3 decimal places.

2

Curve Fitting

The process that we used in the previous problem works in general to solve for the best-approximating coefficients of any (independent) linear combination of functions.

Theorem. Given a set of observations $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ and a set of basis function $f_1, \ldots, f_m$, the least squares approximating function

$$y = \sum_{i=1}^{m} r_i f_i(x)$$

has coefficients $r_1, \ldots, r_m$ which are solutions to the normal equation

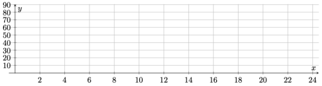
$$\mathbf{A}^\intercal \mathbf{A} \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix} = \mathbf{A}^\intercal \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix},$$

where $\mathbf{A}_{ij} = f_j(x_i)$ for each $1 \le i \le n$, $1 \le j \le m$.

Question 3. Suppose that we wish to find a model for the air temperature in a town. We are given the following average temperature readings for a few months over a two year period:

Month	Temp (°F)	Month	Temp (°F)	Month	Temp (°F)
March 2009	40	May 2009	52	August 2009	84
November 2009	45	January 2010	24	April 2010	34
June 2010	68	September 2010	79	November 2010	48

(a) In order to get an understanding of the shape of the data, plot it on the axes below. Let x represent the number of number of months since the start of 2009 (that is, March 2009 = 3, etc.).



(b) As is expected, the temperature appears to oscillate on a yearly basis, with warmer temperatures in the summer and cooler temperatures in the winter. Because of this, which basis functions would be a natural choice? (**Hint:** You should have a total of 3 basis functions, one of which is a constant function to handle the vertical translation. Also, be sure to account for the period of the oscillation (that is, the length of the repeating cycle) in your basis functions.)

3

(c) Use the Theorem from the previous page to compute the least-squares approximation for your choice of basis functions. Round your coefficients to 3 decimal places.

(d) Suppose that the data from later years shows a slow upward trend in the average monthly temperatures. Suggest another basis function that we can incorporate into our model to allow it to describe this warming behavior.

Measuring the Quality of an Approximation

By choosing different combinations of basis functions, we now have the ability to find many models that approximate the relationships in our sample data. How can we determine which of these models are good? One qualitative technique is visual inspection: does the function we computed in the model lie reasonably close to the data? A quantitative answer to this question comes from calculating R^2, the Coefficient of Determination.

Definition. The coefficient of determination, R^2 is a value (usually) in $[0, 1]$ which describes how much of the error in our approximation can be attributed to the variance of the sample data. When, $R^2 = 1$, our approximation perfectly fits the data, and when $R^2 = 0$, our chosen basis functions do not describe the data. We can calculate R^2 by the formula,

$$R^2 = 1 - \frac{(\mathbf{y} - \mathbf{Ar})^\intercal(\mathbf{y} - \mathbf{Ar})}{(\mathbf{y} - \overline{\mathbf{y}})^\intercal(\mathbf{y} - \overline{\mathbf{y}})},$$

where $(\mathbf{y} - \overline{\mathbf{y}})$ is the vector obtained by subtracting the sample y-mean, $\frac{1}{n}\sum_{i=1}^{n} y_i$, from each entry of $\mathbf{y}$.

4

Curve Fitting

Question 4.

(a) Compute the coefficient of determination for the linear model from Question 1(b). What does this suggest about the quality of the approximation?

(b) Compute the coefficient of determination for the quadratic model from Question 2(b). What does this suggest about the quality of the approximation?

Recap

Given a collection of n sample data points, we can calculate an function that approximates the data by following these steps.

1. Plot the data to observe its general shape (slope, asymptotes, periodic behavior, etc.). Use these observations to select a suitable set of basis functions $f_1, \dots f_m$ to model the data.
2. Use the basis functions to construct an $n \times m$ matrix $\mathbf{A}$.
3. Use the normal equations to solve for the coefficients of the approximation.
4. Evaluate the quality of the approximation by computing the coefficient of determination.
5. If necessary, update the basis functions to improve the quality of the model.

5

Lights out!

In this workshop, we'll be playing a version of "Lights Out", an electronic game from 1995 in which a random subset of lights in a 5×5 grid lights up, and the aim of the game is to figure out how to turn all the lights out by clicking on a preferably minimal subset of the lights. The trick is that by clicking on a light, not only is the light toggled, but so are the (up to) four adjacent lights.

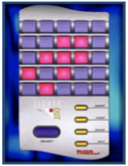
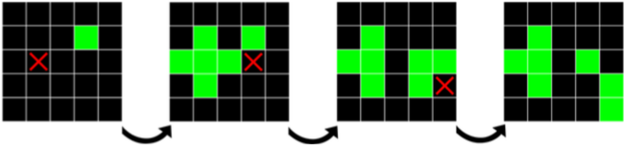


Figure 1: The electronic lights out game (Picture source:`http://matroidunion.org/?p=2160`)

We'll use green and black to denote lights that are on and off respectively. Some examples of game-play are shown below:



https://www.logicgamesonline.com/lightsout/

Projects

- Students complete two application project as well as a reflection essay.
- Each project required students to choose a topic to look into and come up with their own data/information.
- We can expect 260 students in the course.
- This was a first for math at Cornell, only small upper level courses do projects.
- We have a bank of 6 projects.

Projects - Topics

Topic	Semesters	Textbook Section
Simplex Algorithm	Fa21	1.2 Gaussian Elimination
Markov Chains	Fa20, Sp21	2.2 Matrix-Vector Multiplication
Eigenvector Centrality	Sp21	3.3 Diagonalization and Eigenvalues
Networks and Fiedler Sets	Fa20, Fa21, Sp22, Fa22	3.3 Diagonalization and Eigenvalues
Loops in Directed Networks		5.4 Rank of a Matrix
Curve Fitting	Sp22	5.6 Best Approximation

Project 1: Markov Chains

- Students had to find probabilistic data and study a situation with at least three states.
- Individuals need to be capable of going from a state to another and the states must be disjoint.
- Students had worked on examples in discussions: Zombie apocalypse (human, zombie, dead) and Google Page Rank.
- Part of their work was to describe limitations of their results.

We give the rubric with the instructions

Out of 10 points

2 points Presentation: the work is clearly presented and under two pages.

2 points The transition matrix is correct, with explanations.

1 points Correct calculations and description from the initial state.

2 points Correct work finding the steady state vector and interpretation.

3 points A conclusion is present and shows a good analysis.

Sample Projects

Project 1
Marie B.Langlois
Penguin Huddles

Introduction

This project will look into penguin huddles, and the different type of penguins in the huddle: outside penguins "O" the ones on the boundary of the huddle, inside penguins "I" the ones inside the huddle and mover penguins "M" penguins that are relocating to a different spot in the huddle.

Transition Matrix

We look at the data of a huddle of a hundred penguins. At each step, we assume one penguin is moving around the outside (state M), 33 penguins are huddled on the outside (state O), and 66 are huddled on the inside (state I). The assumption that roughly 1/3 of the penguins are on the outside is estimated from two pictures.

The mover penguin always moves to a new a random place on the outside, so the transition probability from M to O is 1. In each new state, a random outsider penguin becomes the next mover, so since there are 33 such penguins, the transition probability from O to M is 1/33, which we approximate by 0.03. An outsider penguin becomes an insider penguin if the mover stops in front of them and blocks them, which also has a probability of around 0.03 (since the mover stops in front of one random outside penguin). We make the simplifying assumption that each time a move happens, it exposes a random penguin from the inside to the outside. Assuming this happens to a random inside penguin means the transition probability from I to O is 1/66, or approximately 0.015. Insider penguins never become movers from one step to the next, so the transition from I to M has probability 0. We compute the remaining transition probabilities so that the columns of the matrix sum to one.

$$P = \begin{array}{c} \\ O \\ I \\ M \end{array} \begin{array}{c} \begin{array}{ccc} O & I & M \end{array} \\ \begin{bmatrix} 0.96 & 0.015 & 1 \\ 0.03 & 0.985 & 0 \\ 0.03 & 0 & 0 \end{bmatrix} \end{array}$$

Finite State Calculation

If you are a cozy inside penguin, what are the odds of still being inside after 10 iterations? This can be answered by taking the vector **s**_0=[0 1 0]^T and calculating

s_(10)=P^(10) **s**_0=[0.12   0.8766   0.0033]^T.

This calculation was done using Sage CoCalc. An inside penguin has an 88% chance of still being inside after 10 iterations.

Steady State Calculation
We want to find **s**=[x y z]^T such that P**s**=**s**, we can do so by finding a solution to (P-I)**s**=0 such that x+y+z=1. I used Sage CoCalc to reduce P-I and obtained that it had the identity matrix as its reduced form. Therefore the steady state vector is x=y=z=0, which is not possible in this situation. Mathematically speaking the situation does not stabilize.

Conclusions and Limitations

In reality penguin huddles do stabilize, but since our model oversimplifies the situation, we did not obtain this result. This model assumes that a penguin will move at each iteration, but eventuality there won't be any moving penguins. The ration of inside/outside penguins changes through the iterations, but this was ignored. The model also assumes that all outside penguins have the same probability of becoming mover penguins, that is not the case there are many environmental factors that affect this.

Sources

I have used the following two articles:

- *Math of the Penguins*, Susan D'Agostino, quantamagazine, August 17, 2020, url: https://www.quantamagazine.org/math-of-the-penguins-20200817/
- *Modeling Huddling Penguins*, Aaron Waters, Francois Blanchette and Arnold D. Kim, November 16, 2012, url: https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0050277

Sample Projects

Marie B.Langlois
Math 2210
Project 1 - Markov Chains

Introduction

This project will expose in a general way the yearly retention and recidivism rates for individuals who are in prison or have gone to prison. The three stages will be: being in prison (P), being out (O) and being dead (D).

Sources

The following websites and articles have been used for data:

1. *Long-Term Recidivism Studies Show High Arrest Rates*, Matthew Clarke, Prison Legal News, May 3, 2019. Link to the article.
2. *Mortality in State and Federal Prisons, 2001-2016 – Statistical Tables*, E. Ann Carson, Mary P. Cowhig, U.S. Department of Justice, February 2020. Link to the article.
3. *Reentry Trends In The U.S.*, Bureau of Justice Statistics. Link to the website.
4. *Mortality in the United States, 2018*, National Center for Health Statistics, Center for Disease Control and Prevention. Link to the website.

Building the Transition Matrix

An individual who is out of prison has a 45% probability of going back in the next year [1]. From [3] we have that release rate is 33%. According to [2] the mortality rate in prison is 252/100,000, I used the federal prisoner mortality rate from 2016, as it is in between state and federal rates over the years. From [4], we get that mortality rates are higher for the general population and are of 863.8/100,000. This gives the following transition matrix:

$$P = \begin{array}{c} \\ P \\ O \\ D \end{array} \begin{array}{c} \begin{array}{ccc} P & O & D \end{array} \\ \begin{bmatrix} 0.66748 & 0.45 & 0 \\ 0.33 & 0.541362 & 0 \\ 0.00252 & 0.008638 & 1 \end{bmatrix} \end{array}$$

the other values were obtained by making the columns add to 1.

Initial State Calculation

If an individual is out of the criminal system on a given year, what is the probability that they are still out in 5 years? We start with the initial state $\mathbf{s}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, we can calculate $\mathbf{s}_5$ in the following way:

$$\mathbf{s}_5 = P^5\mathbf{s}_0.$$

Using Sage CoCalc for calculations, I obtained $\mathbf{s}_5 = \begin{bmatrix} 0.5614 & 0.4089 & 0.02 \end{bmatrix}^T$. I also calculated $\mathbf{s}_{20} = \begin{bmatrix} 0.5202 & 0.3785 & 0.1012 \end{bmatrix}^T$, the probability of being out did not decrease as much as I expected.

Steady State Calculation

To find the steady state, that is $\mathbf{s}$ such that $P\mathbf{s} = \mathbf{s}$, I solved the homogeneous system $(P - I)\mathbf{s} = \mathbf{0}$. I used Sage Cocalc to obtain the following reduced echelon form:

$$\begin{bmatrix} 1 & 0 & -0.0000 \\ 0 & 1 & -0.0000 \\ 0 & 0 & 0 \end{bmatrix}$$

I have interpreted -0.0000 as a very small negative number. This gives the parametric solution for $\mathbf{s} = \begin{bmatrix} x & y & z \end{bmatrix}^T$, with $x = \alpha t$, $y = \beta t$, where α and β are really small positive numbers, and $z = t$. Since $x + y + z = 1$, we will have that z is very close to 1, which seems reasonable as eventually all individuals will die.

Conclusions and Limitations

This model is definitely oversimplifying many factors that affect this situation. Overall, given the difference between $\mathbf{s}_5$ and $\mathbf{s}_{20}$, the probabilities of being in and out of prison seem stable. Since the transition matrix has a 1 in one of its columns, the model will stabilize to that state. I was surprised by seeing that the mortality rate is lower in prison than out, this might be because there is no universal healthcare in the U.S., or ill incarcerated individuals being released before passing away. The model does not reflect the fact that the probability of dying increases as people get older. The data used to build the transition matrix did not consider various factors that affect releases and recidivism such as race, age, gender, type of crime, etc.

The most popular topics

What topics do students choose when given freedom in a math project?

- 22 projects on COVID-19
- 15 projects on Elections
- 14 projects on the weather
- 7 projects on income mobility
- 6 projects on stock prices
- 6 projects on alcohol disorders

Some more original ones

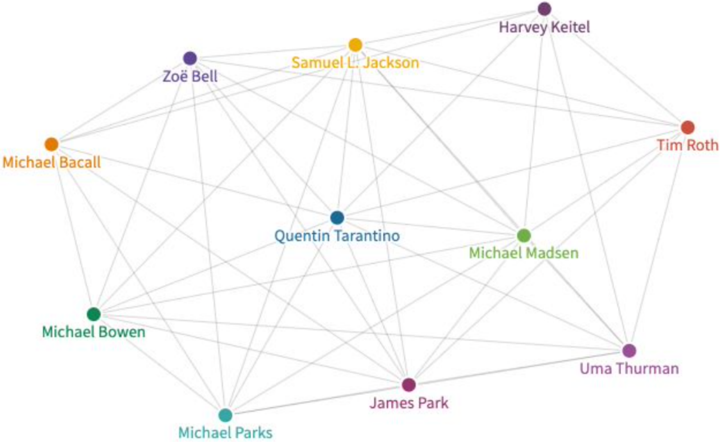
What topics do students choose when given freedom in a math project?

- Sleep cycle of mice
- Genetics of being a red head
- Evolution stages of the milky way
- Smart phone brand retetention
- Mario Kart
- Career progression of government officials

Project 2: Graph Theory

- We have two different versions:
  - Eigenvector centrality.
  - Fiedler set.
- Students must come up with their network and do some analysis using matrix calculations.
- Their graph must have 6 to 15 nodes.
- They are encouraged to use software for calculations (Wolfram Alpha, Symbolab, Julia).

Tarantino Movies



European Railway System

Color

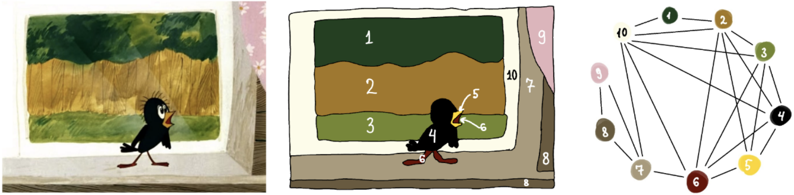


Figure 1: Initial image, color-simplified version and the network of colors

$$maroon > ochre > beige > (green, black) > yellow > light{-}brown > dark{-}green > (brown, pink).$$

Math 2210 - Code Demos

Alex Townsend developed MATLAB demonstrations on the following topics:

- Introduction to Linear Systems
- Polynomial Interpolation
- Balancing Chemical Reactions
- Linear Systems with Multiple Right Hand Sides
- Column Row Major Ordering
- Multiplying a Vector by a Matrix Product
- Finding Roots via Eigenvalues
- Compressing Flags
- Singular Value Decomposition for Image Compression

MATH 4310 - ALI

- Creation of in-class worksheets.
- Video assignment.
- Project:
    - Pick a scientific article, summarize it and reflect on it.
    - We allowed group submissions.
- The course is going through its third iteration of the restructure.

MATH 4310 - Worksheets

- Between 30% to 50% of class time is spent on worksheets.
- When possible we try to have the students come up with the results (IBL).
- They have taken LA before and they are expected to be comfortable with proof writing.
- We are using *Linear Algebra Done Right* by Sheldon Axler (free access through the library).
- Book only works over $\mathbb{C}$ and $\mathbb{R}$, we made sure to include $\mathbb{Z}/n\mathbb{Z}$ (and $\mathbb{Q}$).

Definition of a field

Introduction to Fields MATH 431
01/24 Workshee

In this course we will revisit your knowledge of Linear Algebra in a more general way and prove that ou previously computational tools work in an abstract setting. During class, you will often be asked to come u with definitions, conjecture and proofs, as a group we will work perfecting these.

1. Get to know your group, share the following:
   - preferred name and pronouns
   - favorite breakfast food
   - if you feel comfortable, exchange emails, this might be helpful later in the term

2. The goal of this worksheet is to think of sets of mathematical S objects that we use and operations o these. Start with the real numbers with the operations of addition and multiplication $(\mathbb{R}, +, \cdot)$. Lis properties of these:
   - There exist 0 such that for all $a \in S$, $0 + a = a$.
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3. If you have not done so above, define subtraction and division using $(+, \cdot)$.

Introduction to Fields MATH 4310
01/24 Worksheet

4. Keeping in mind the properties you listed on the previous page, this question will have you think of other mathematical objects with the same or different operations.
   (a) List sets other than $\mathbb{R}$, that respect all above properties.

   (b) List sets that do not respect certain properties. Write down the set and the property(ies) that it fails.

For each property, can you find a set that fails it?

Instructor Notes

Week 2 MATH 4310
01/31 Instructor Notes

Learning Objectives: (1B, 1C) by the end of the week students will be capable of

- Define and give examples of vector spaces.
- Prove properties about vector spaces.
- Define and give examples of subspaces.
- Work with sums and direct sums of subspaces.

Instructor Notes:

There is one worksheet for the week, worksheets are always meant to be completed in groups. Here is a rough plan for the week.

Monday:

- Give the abstract definition of vector spaces and list a few examples.
- Problems 1-3 on the worksheet.
- Prove a simple fact about vector spaces, discuss proof writing along the way. (For example $0 \cdot v = \mathbf{0}$ for all $v \in V$.)
- Problems 4,5 on the worksheet.
- Tell students to read 1B before next class.

Instructor Notes

Wednesday:

- Define subspaces and prove that the condition for checking for a subspace is correct.
- Problems 6,7 on the worksheet.
- Allow plenty of time to recap problem 6.

Friday:

- Define the sum of subspaces and prove that the sum is the smallest subspace containing two subspaces, or prove another theorem from this section.
- Define direct sums.
- Problems 8,9 on the worksheet.
- Tell students to read 1C before next class.

Direct Sums

7. True or false, if true, prove it and if false, give a counterexample. For U and W subspaces of V:
   (a) $U \cap W$ is a subspace of V.

   (b) $U \cup W$ is a subspace of V.

8. If U is a subspace of V, what is $U + U$?

9. (a) Take a simple vector space V and 2 subspaces U, W such that $V = U \oplus W$.

Vector Spaces and Subspaces MATH 4310
Week 2 Worksheet

   (b) What is $U \cap W$?

   (c) Take $0 \in V$, write it as a sum of vectors in U and W, in how many ways can you do this?

   (d) Do you think that your results generalize to any U, W, V?

   (e) Do you think that your results generalize to $V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$, where the U_is are subspaces of V?

Inner Products

1. Determine whether the following functions are inner products.
   (a) Let $V = \mathbb{R}^2$ and define $\langle (a, b), (c, d) \rangle = ac - bd$.

   (b) Let $V = \mathbb{C}^n$, for $v = (w_1, \ldots, w_n)$, $u = (z_1, \ldots, z_n)$ and define $\langle v, u \rangle = a_1 w_1 \overline{z_1} + \cdots + a_n w_n \overline{z_n}$, wher $a_i \in \mathbb{C}$.

   (c) Let $V = M_2(\mathbb{R})$ and define $\langle A, B \rangle = \mathrm{tr}(A + B)$.

2. (Don't look at the textbook yet). The goal of this question is to prove:

Theorem: Cauchy-Scharz Inequality Suppose $u, v \in V$. Then

$$|\langle u, v \rangle| \leq ||u|| \, ||v||.$$

Hint: use the orthogonal decomposition and the Pythagorean Theorem.

Math 4310 - Video Assignment

- Worth 5% of their grade, graded on reasonable completion.
- Students could work in groups of up to 3.
- Record 5-10 min video solution to a homework problem.
- Students had to sign up for the problem they would record a solution.
- Videos needed an audio and visual component.
- They were not graded on the correctness of the solution for the video part.

Math 4310 - Project

- Worth 5% of their grade.
- They were given the rubric with the instructions.
- They had to find a scientific article that uses linear algebra and summarize it.
- Group submissions were allowed, project were to be at most of length $n + 1$.
- Half the submissions we on machine learning.

Group Activity

In your breakout rooms:

- Think of how you could incorporate projects, in a self-guided way.
- Students should have some freedom choosing a topic.

Group Activity

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ALI - Other Courses

Under both grants, the following courses have been transformed:

- Math 1106, Modeling with Calculus for the Life Sciences.
- Math 1110, Calculus I.
- Math 1300, Mathematical Explorations.
- Math 3040, Prove it!

Math 1106

- Changed from Calculus for the Life and Social Sciences to Modeling with Calculus for the Life Sciences.
- Using the textbook *Modeling Life* by Garfinkel, Shevtsov and Guo.
- Only offered in the Spring, two sections of about 115 students.
- Existing class and recitation materials.
- Class time recently changed from 2 to 3 hours a week, we will readapt materials this summer.

Math 1110

- Materials exist for flipped classes, that is pre-class activities and worksheets.
- Our first year using the textbook *Active Calculus* by Matthew Boelkins.
- Taught in sections of roughly 30 students.
- Focus is on concepts and explanations.
- Homework is a mix of WeBWorK and a few problems where writing is an important part of the answer.

CALM Website

https://e.math.cornell.edu/sites/activelearn/index.html

CALM

Cornell Active Learning in Mathematics

What is Active Learning?

Course Materials

Meet Our Team

Publications

Thank you for listening.