Using Slopes to Enhance Learning in Differential Equations

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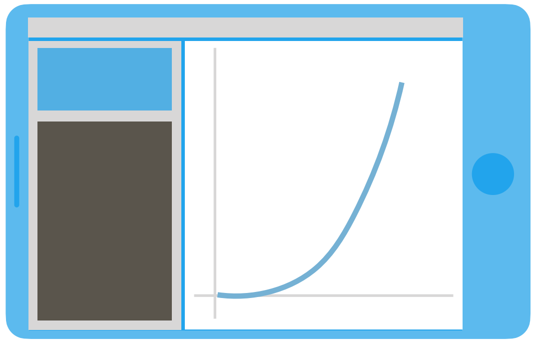
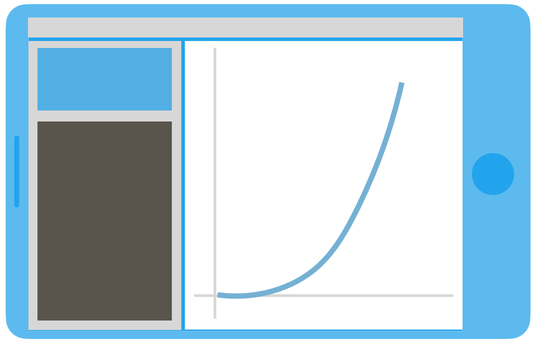
Check out the apps at
http://www.slopesapp.com
http://www.wavespdeapp.com



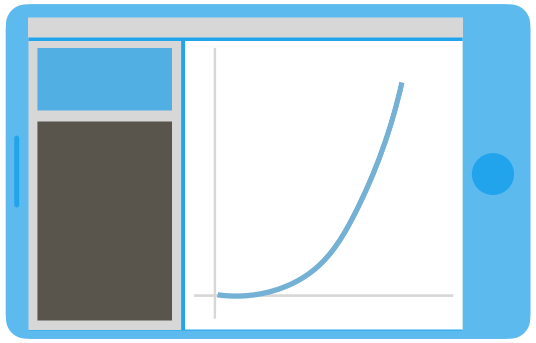
Class Approach

* Physical assumptions
* Mathematical expressions (DEs)
* Think before you solve
    * Equilibrium solutions
    * Graphical analysis
        * Slopefields, Phase Planes, etc
* Analytical Solutions



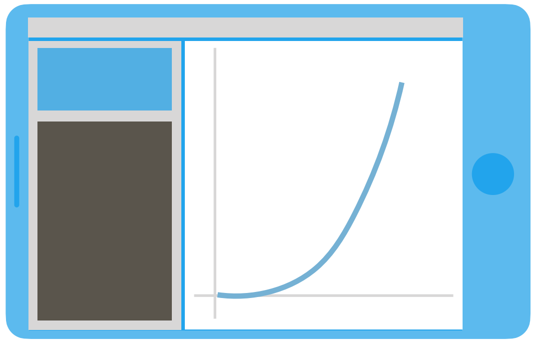
Obstacles to Visualization

* Professional Software
  * Maple/Mathematica/MATLAB
  * Learning Curve/Syntax
  * Expensive
* Free Software
  * dfield/pplane
    * Less support for Java
  * GeoGebra/Desmos
    * Not designed for Diff Eq
* Most produce static images



App Information

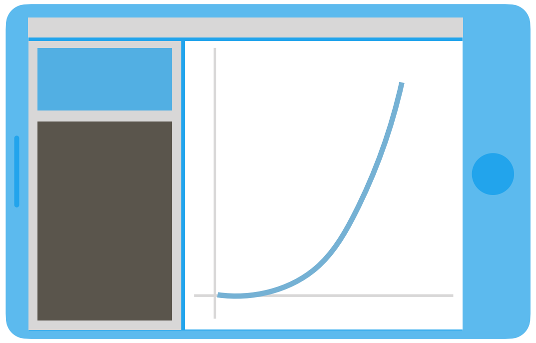
* Languages: **Swift and Kotlin**
* Platforms: **iOS and Android**
* http://slopesapp.com (ODE)
  * iPad Release: **Nov 2016**
  * iPhone Release: **July 2017**
  * Macs with M Chips: **Nov 2020**
  * Android Release: **Jan 2020**
* http://wavespdeapp.com (PDE)
  * iOS Release Date: **June 2019**



Why Mobile Devices?

* Portable
* Comparatively large screen
* Tactile interface

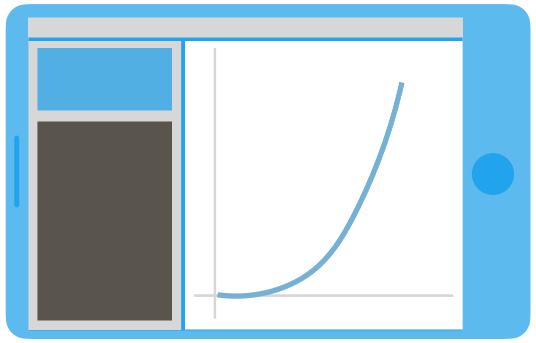
The Role of iPads in Constructing Collaborative Learning Spaces (Fisher, Lucas and Galstyan, 2013)



Study Framework

* Inquiry and active engagement are important in mathematics education (IODE, CBMS)
* Mathematical learning is conversational (Ernest, 1994)
* Tools (i.e. Slopes) create collaborative classroom spaces

Using Slopes to Enhance Learning in Ordinary Differential Equations (K. Lucas and T. Lucas, 2022)



Bifurcation

A **bifurcation** occurs when a small change in a parameter value leads to a qualitative change in the long term behavior of the solution to a differential equation.

Local Ecology

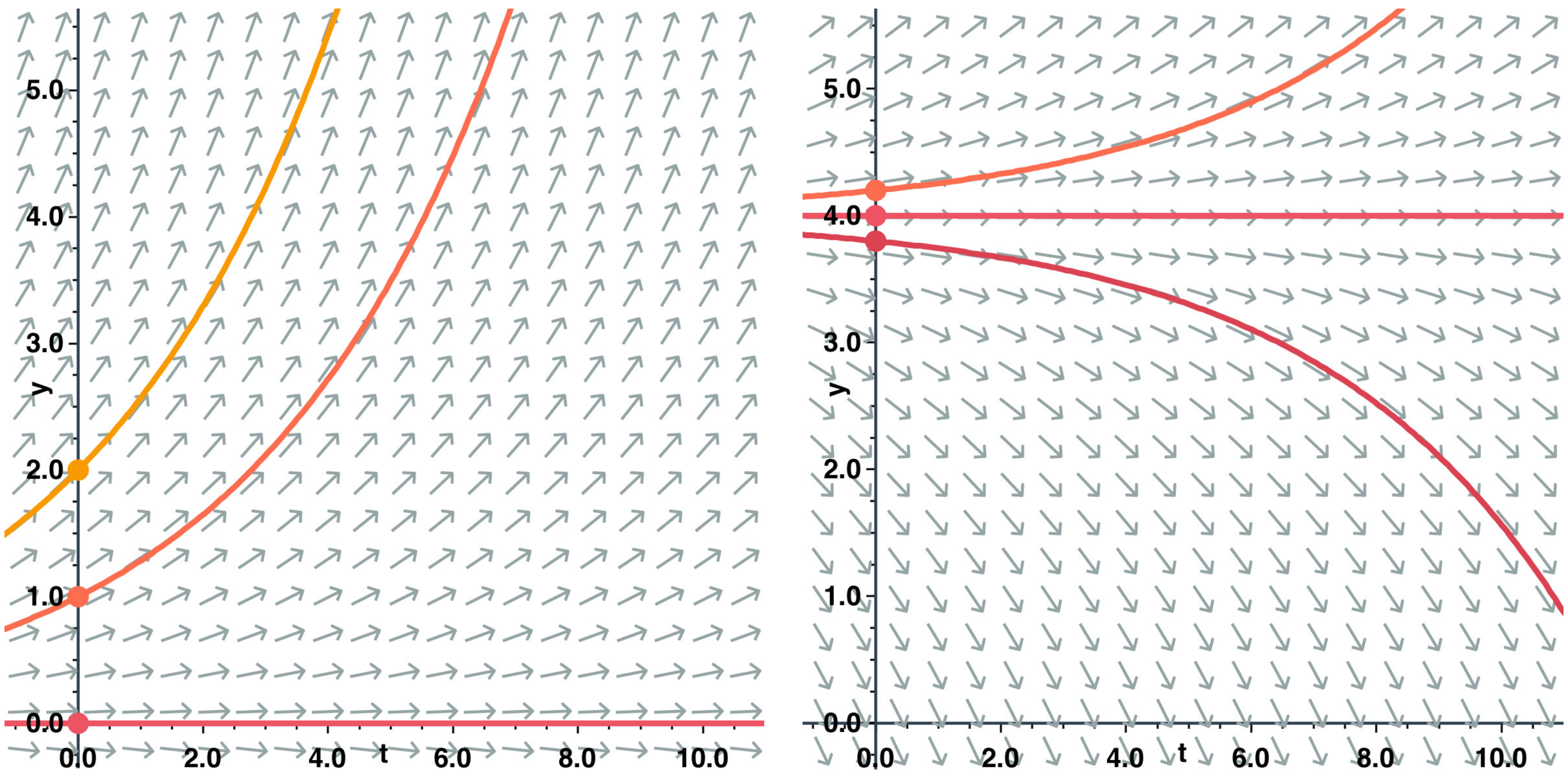
California Newt - *Taricha torosa*

Invasive Crayfish

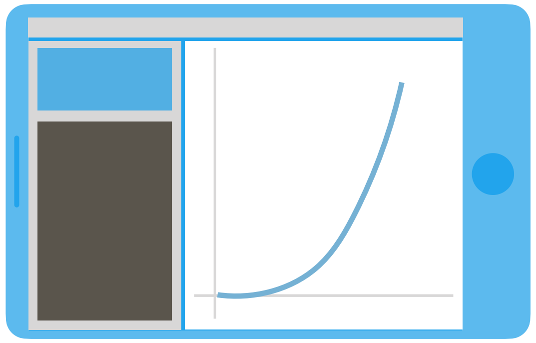
Procambarus clarkii

Crayfish Models

* Exponential Growth/Decay: **y' = ay**
* Exp Growth w/ Removal: **y' = ay - c**



* Same graph with shifted equilibrium



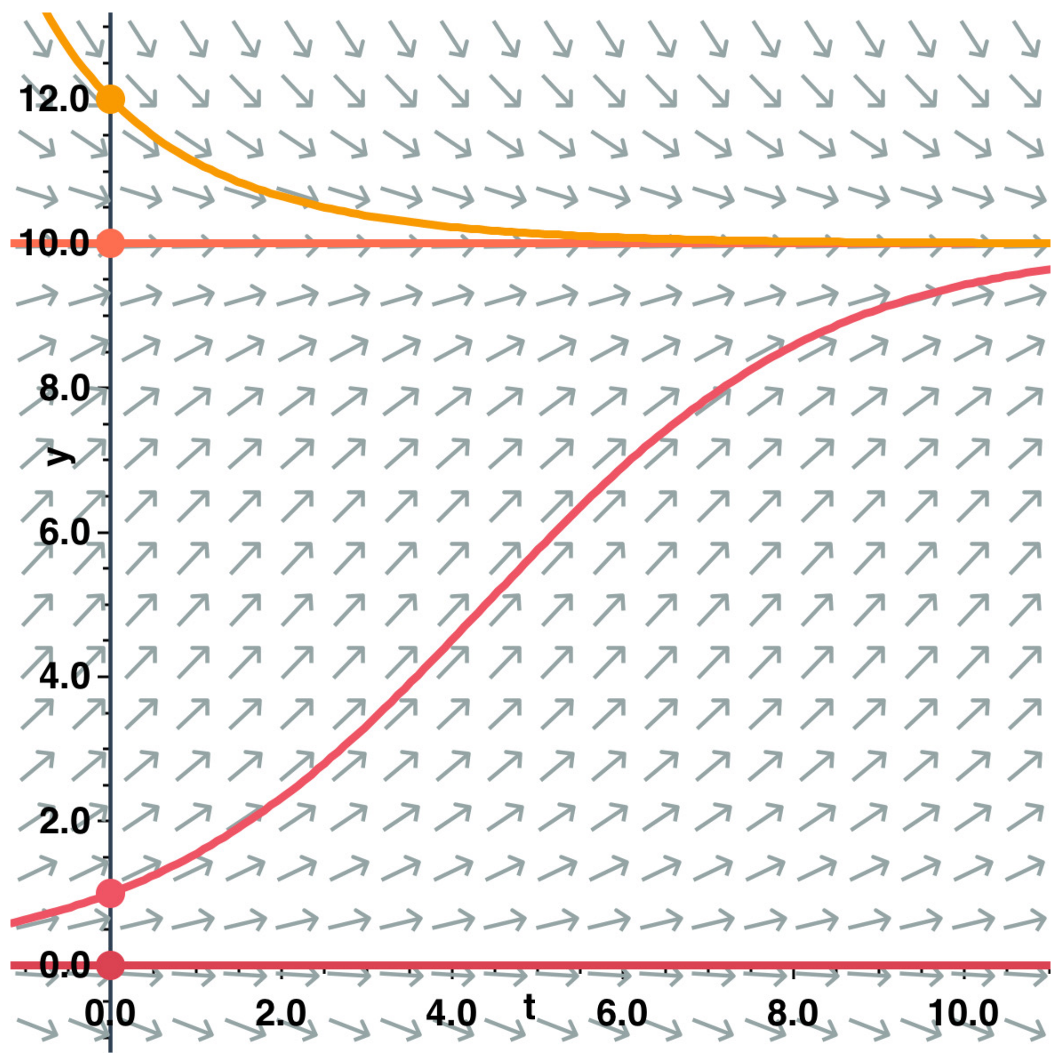
Class Activities

Observations while Using Slopes

- What do the arrows represent?
- What patterns do you observe?
- How do those patterns relate to the model and differential equation?
- What is the relationship between the arrows and the solution curves?
- Describe the behavior of the solutions for various initial values.
- Are there any equilibrium solutions?
- Use the plot to describe the stability of the equilibrium.
- What happens as you vary parameters?

Crayfish Models

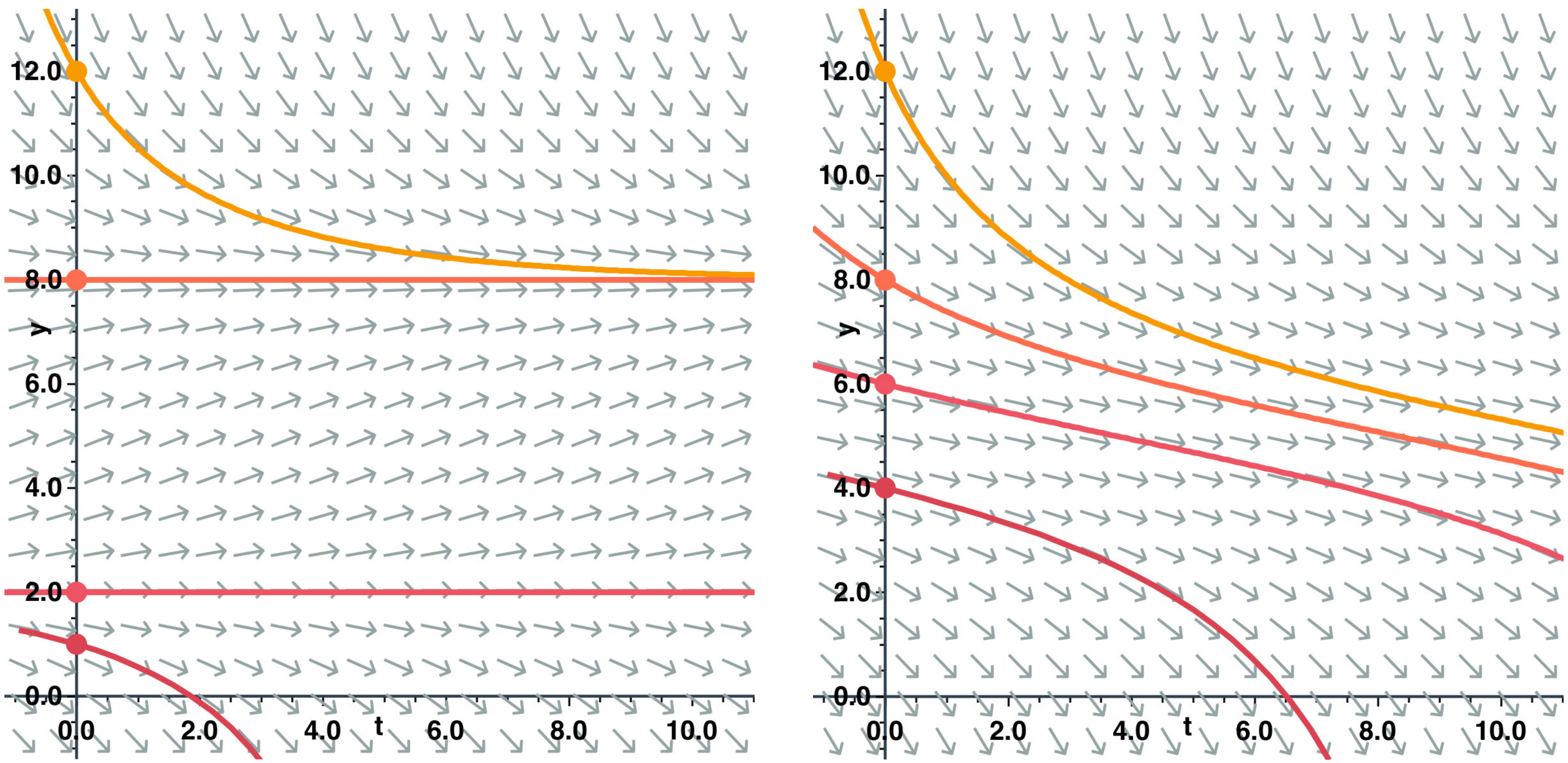
* Logistic Growth: **y' = ay(1-y/b)**



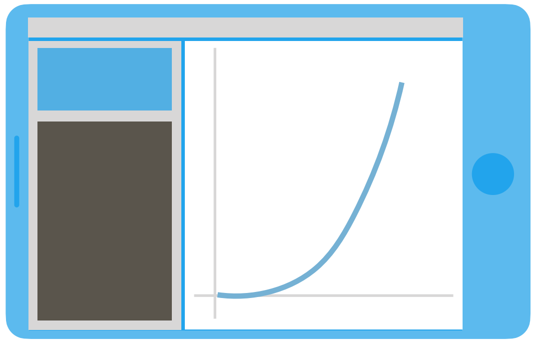
* One stable, one unstable equilibrium

Crayfish Models

- Logistic Growth with Constant Removal: **y' = ay(1-y/b) - c**

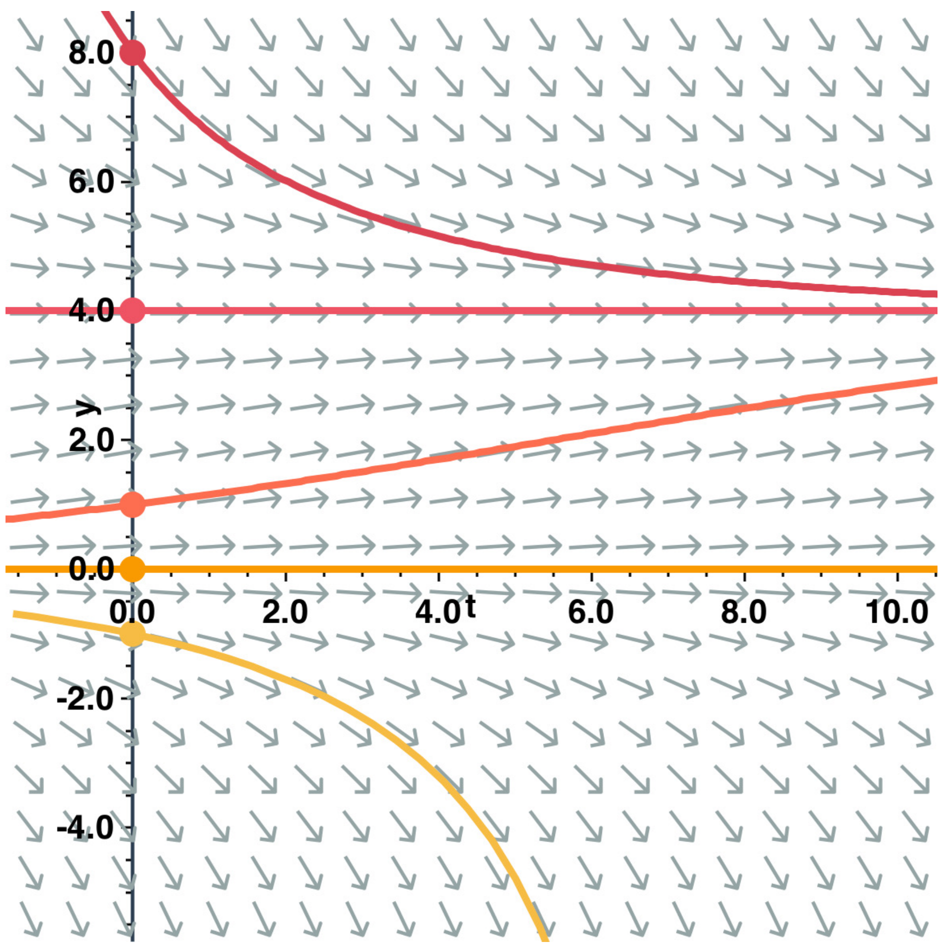


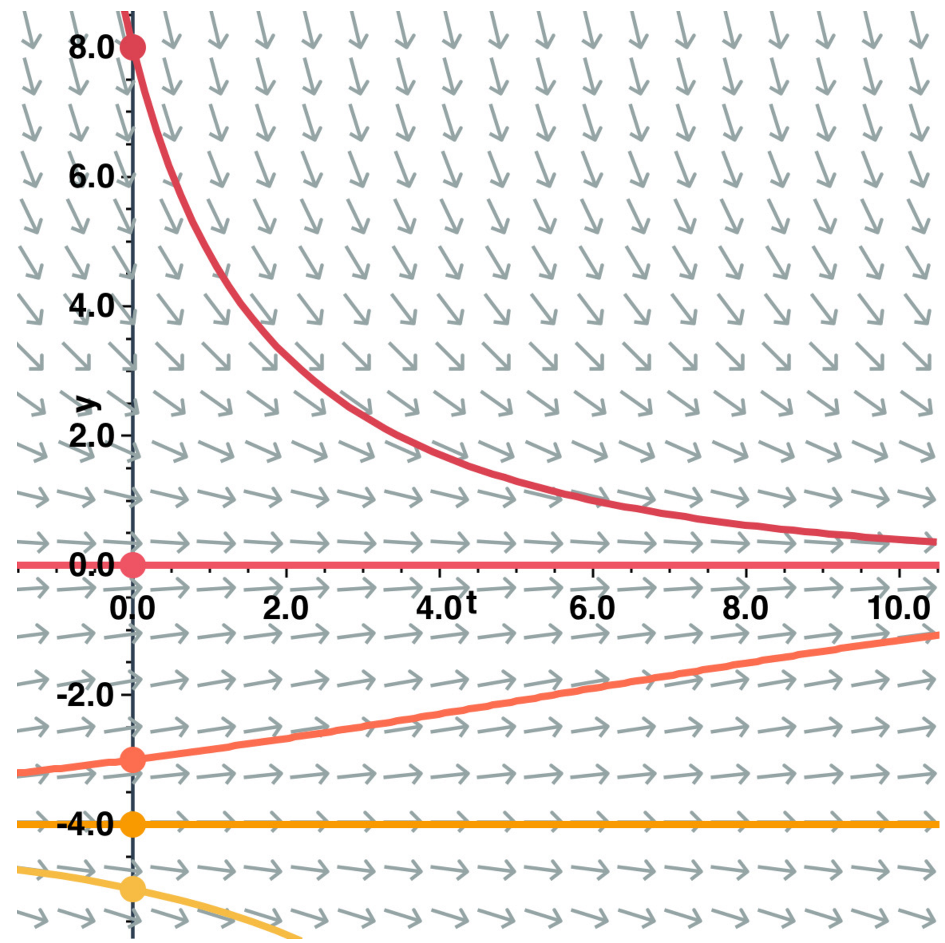
- Increase c, 2 equilibria become 0



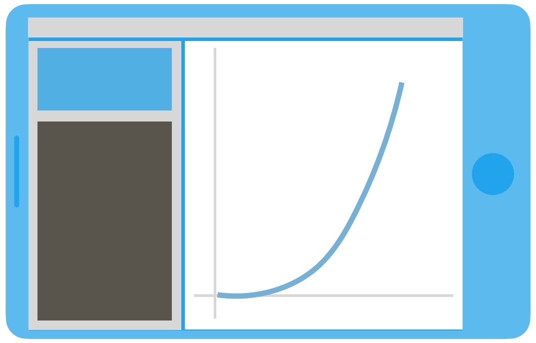
Crayfish Models

- Logistic Growth with Proportional Removal: $\mathbf{y' = ay(1-y/b) - cy}$





- Increase c, equilibria switch stability



Class Activities

1. **Logistic Growth:** Recall the following population model

$$\frac{dy}{dt} = ay\left(1 - \frac{y}{b}\right)$$

Use the slopefields activity to examine the model with $a = 0.5$ and $b = 10$.

Discuss any observations can you make about the slopefield and solution curves.

- What do the arrows represent?

  Arrows represent the slope

- Add some representative solution curves to plot. What is the relationship between the arrows and the solution curves?

  Connect arrows, tracing the solution at particular starting point.

- Are there any equilibrium solutions? Use the plot to describe the stability of the equilibrium solutions.

  $y = 0, 10$

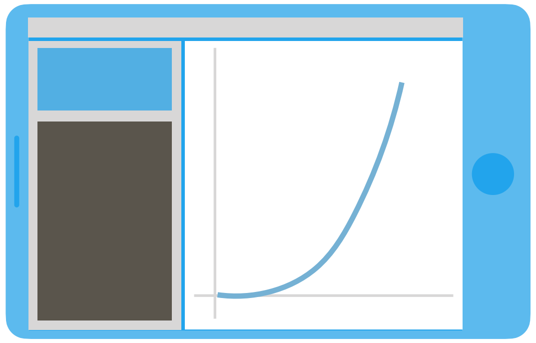
  0 → unstable

  10 → stable

- Interpret the solutions in the context of a population model.

  0 ⇒ unstable 10 ⇒ stable

  Start below 10 and ≠ 0 → 10 Above 10 → 10



Class Activities

Consider the following modification of the previous model

$$\frac{dy}{dt} = ay\left(1 - \frac{y}{b}\right)\left(\frac{y}{c} - 1\right)$$

Use the slopefields activity to examine the model with $a = 0.5$, $b = 10$, $c = 4$.

Plot the slopefield and some representative solution curves. Discuss any observations.

- Describe the behavior of solutions for various initial values.

  0 < y' < 4 → 0   y > 10 ⇒ 10

  4 < y' < 10 → 10

- Are there any equilibrium solutions? Use the plot to describe the stability of the equilibrium solutions.

  y' = 0, 4, 10   0 → stable   10 → stable

  4 → unstable

- Interpret the parameter c in the context of a population model.

  c is the threshold of which values above it will grow to carrying capacity and if below, will decay to 0.   y' < c → 0

  y' > c → b

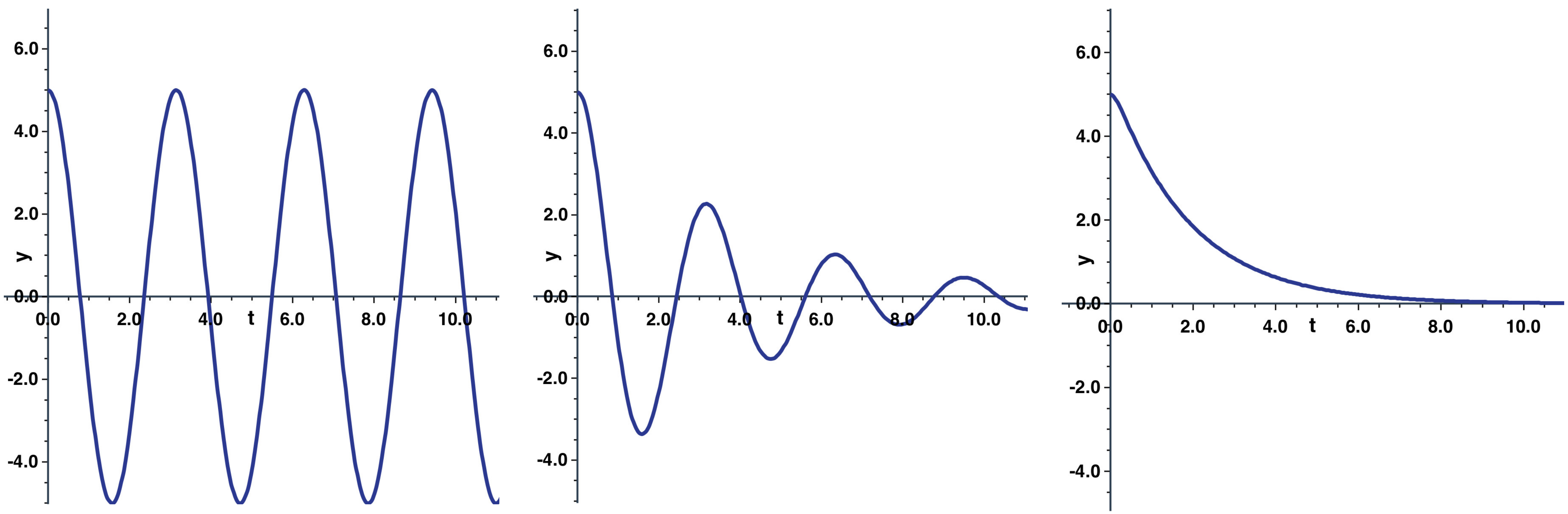
Class Activities

Below is a discussion about the equilibrium solutions in the threshold model.

JAMES: So it would be anything greater than four.
ALEX: Ten, four, and zero.
JAMES: And then four is unstable...It's like the opposite of a carrying capacity.
DOMINIC: It's a threshold, right?
JAMES: Oh yeah, it is. So if you're above the threshold, then you go to the carrying capacity.
DANIEL: If you're below, you're going to go to zero.
ALEX: It's like the minimum population you need to not die out.

Damped Oscillations

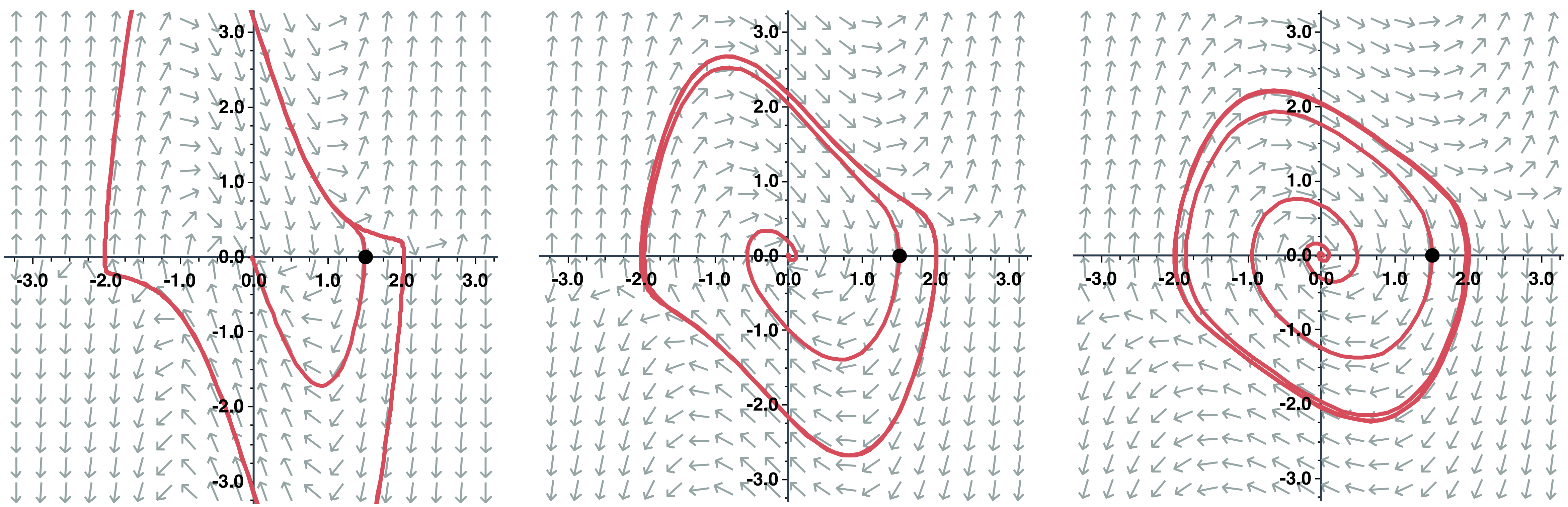
* Mass-Spring System: **ay''+by'+cy=0**



* Transitions from undamped to underdamped to overdamped as b increases

Van der Pol Oscillator

* x'=y,   y'=-x+a(1-x^2)y

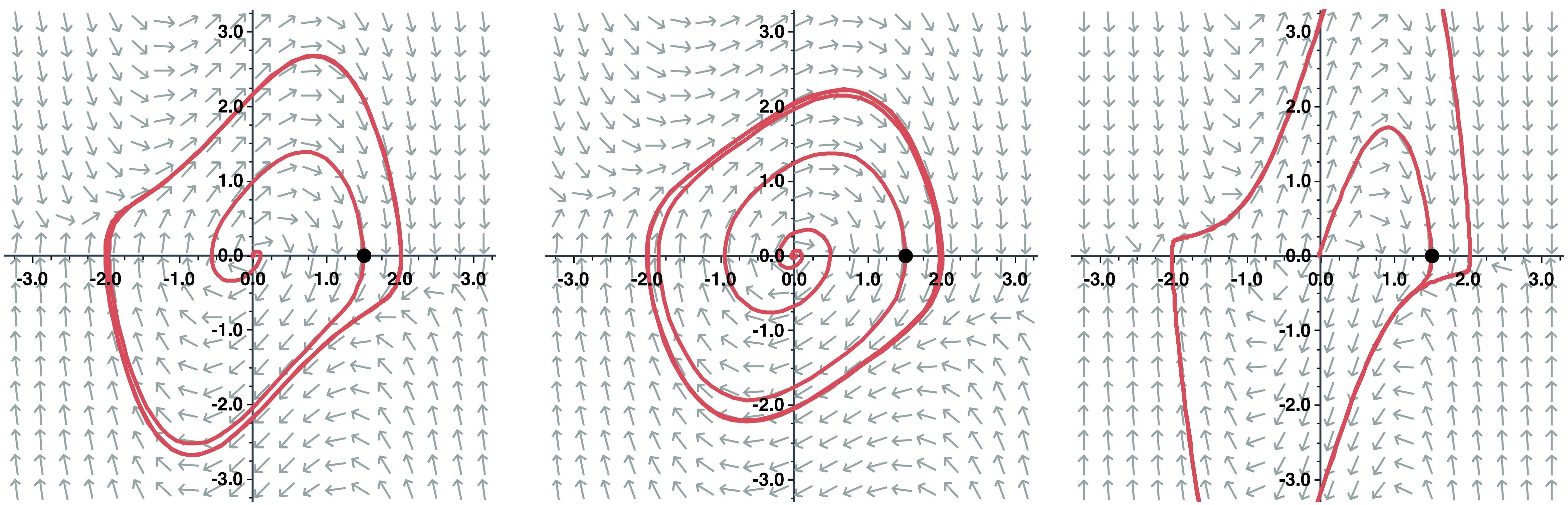


a=-3   a= -1   a=-0.5

* Equilibrium point changes as a increases

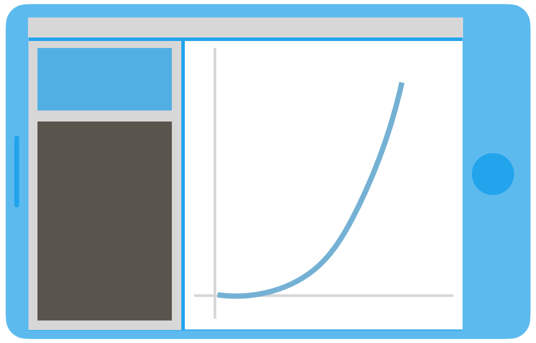
Van der Pol Oscillator

- $x' = y, \quad y' = -x + a(1 - x^2)y$



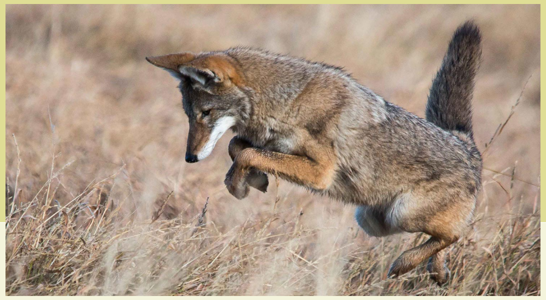
a=0.5 a=1 a=3

- Equilibrium point changes as a increases



Semester Projects

- Main Goal: Understand and analyze a mathematical model using techniques learned in class
  - Slopefields/Phase Planes, Equilibrium Analysis, Numerical/Algebraic Solutions, ...
- Teams of 3-4 Students
- Final Poster Presentation
  - Judged by math and science faculty
- Images provided by Slopes



Mountain Lions vs. Deer
Three Models Examining Predator Prey Dynamics

Background

Over the last 16 months, Pepperdine has issued 17 warnings regarding mountain lions spotted on campus. In an effort to understand the dynamics behind this rise in sightings (and the Malibu ecosystem in general), we use predator/prey systems of differential equations.

Within California, mountain lions feed primarily on deer, and deer are preyed on primarily by mountain lions. Given the omnipresent nature of deer on campus, this likely extends to our local ecosystem. We aim to better understand this relationship, we compare three predator/prey models with increasing complexity.

Basic Model and Coefficient Estimates

In its most nascent form, our model includes two species (mountain lions and deer), and models deer using exponential growth. The variables x and y represent deer (prey) and mountain lion (predator) populations, respectively. The equations are displayed below.

$$x' = ax - bxy$$
$$y' = -cy + dxy$$

Coefficients:

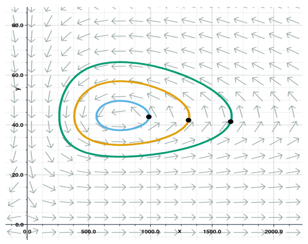
a	Rate of growth without predation.
b	Rate at which predation (interactions) decreases deer population.
c	Rate at which mountain lions die without prey.
d	Rate at which predation (interaction) increases mountain lion population.

While the true values of some parameters are unknown, zoological research can be used to guide many of these choices:

- Deer's maximum growth rate is estimated using reproduction statistics.
  - At any given time, 66% of female deer are pregnant. Deer have an average litter size of 1.9, are pregnant for about 203 days, and exhibit balanced sex ratios.
  - If we observed a deer population and returned 203 days later, we would therefore expect to see 62.7% more deer. Therefore, without external constraints the population will grow according to the equation $x = ce^{0.875t}$, with c denoting the starting population, and t denoting change in time (in years).
- The location of equilibriums is estimated using food intake requirements.
  - Should the growth in the deer population produce exactly the amount of meat required to sustain the mountain lion population, no change should occur in either.
  - The average mountain lion requires 6.57 pounds of meat per day to survive. As the average mule deer weighs 177 pounds, with at most 133 pounds being edible biomass, a mountain lion's survival requires 0.049 deer per day, which is 18.0 per year.

First Model Behavior

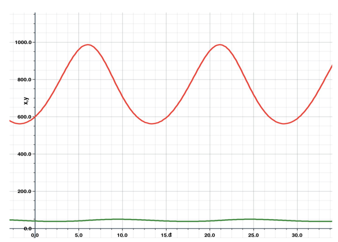
The phase plane has two equilibrium points: a saddle point at (0,0) and a center at (c/d, a/b). Only in cases of complete extinction and at (c/d, a/b) are both populations at rest. A phase plane is shown below, along with a graph of each population. Coefficients were informed by the research described above. Except in the case when an initial value is zero, populations continuously orbit around the center.



a = 0.875, b = 0.02, c = 0.2, d = 0.000265

Initial Conditions: Deer = 600, Mountain Lion = 40

Equilibrium points: (755, 44), (0,0)



Jacobian at (0,0): $\begin{bmatrix} 0.87 & 0 \\ 0 & -0.2 \end{bmatrix}$, *λ1= 0.87, λ2= -0.2 (unstable, saddle)*

Jacobian at (755, 44): $\begin{bmatrix} 0 & -15.1 \\ -0.012 & 0 \end{bmatrix}$, *λ1= 0.417i, λ2= −0.417i (unstable, non−generic, confirmed as center using numerical methods.)*

Logistic Growth and Ratio Dependence

Our second model introduces the concepts of logistic growth and density dependence. The equations are displayed below.

$$x' = ax\left(1 - \frac{x}{k}\right) - \frac{bxy}{(1 + fx + y)}$$
$$y' = -cy + \frac{dxy}{(1 + fx + y)}$$

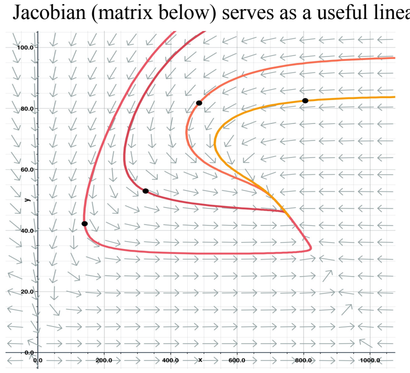
Logistic growth is introduced by the addition of (1-x/k) to the part of the equation controlling deer growth. This takes resource scarcity into account, and ensures that the deer population do not expand beyond its carrying capacity (k). The importance of this feature is highlighted by a potential behavior observable in the first model: without mountain lions the deer population will expand infinitely. While California's natural carrying capacity for deer is an unexpectedly contentious topic, the deer population peaked at approximately 2 million before declining substantially to its current level of 540,000.

Density dependence is introduced by the denominator now beneath both interaction terms. The effect of this change is best explained by an example. Say that an environment has 100 deer and 1 mountain lion. Say that a second environment has 10 deer and 10 mountain lions. Under our first model, both interaction terms end up the same: 100 times some interaction coefficient. This is obviously a departure from how deer and mountain lions interact in reality. Thus, the new model makes the effectiveness of predation dependent on this ratio between species. The most effective ratio can be determined using the value the coefficients (in this case just f, as one can always be omitted). For example, if f = 1, the second environment would yield a larger interaction term under the new model.

To ensure comparability between models, values of coefficients representing biological constants (a, c) remain unchanged from the previous model, and interaction coefficients (b, d) are appropriately scaled to adjust for the new terms.

Second Model Behavior

The phase plane of the second model is shown below. As the model is not linear, the Jacobian (matrix below) serves as a useful linear approximation.



Parameters:

a = 0.875, b = 1.8, c = 0.2,

d = 0.1, f = 0.44, k = 1000

Equilibrium points: (757, 44), (0,0), (1000, 0)

Jacobian:

$$\begin{bmatrix} a - \frac{2ax}{k} - \frac{by(y+1)}{(fx+y+1)^2} & \frac{-bx(fx+1)}{(fx+y+1)^2} \\ \frac{dy(y+1)}{(fx+y+1)^2} & -c + \frac{dx(fx+1)}{(fx+y+1)^2} \end{bmatrix}$$

Using this Jacobian, the equilibrium points can be categorized. The Jacobian for the only equilibrium where both species survive is shown below, along with the useful indicators it provides.

$$J = \begin{bmatrix} -0.45896 & -3.1775 \\ 0.0014 & -0.2347 \end{bmatrix} \quad \lambda_1 = -0.257 \quad \lambda_2 = -0.437 \quad tr(J) = -0.694 \quad det(J) = 0.112$$

This makes (757, 44) a nodal sink (stable). Using the same techniques, (0,0) and (1000, 0) can both be categorized as saddle points.

When initial populations are non-zero, they will eventually sink into a single equilibrium of 757 deer and 44 mountain lions. Unlike the previous model, the equilibrium is always approached (nodal sink) rather than circled around (center).

Adding a Third Species

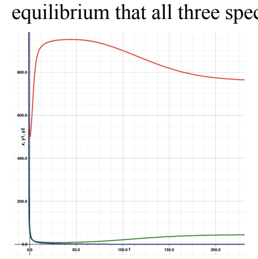
When compared to empirical data, it becomes clear that both models discussed though this point are incomplete. Just 5,000 wild mountain lions roam California—well below what our two-species model predicts. While a wildlife policy of suppressing dangerous mountain lions while protecting deer from predation is likely the largest culprit, it is also worth considering other interactions within our ecosystem. Our third and final model maintains the concepts discussed in its antecedent while adding a third species: coyotes. The equations are displayed below, with y_1 denoting mountain lions and y_2 denoting coyotes.

$$x' = ax\left(1 - \frac{x}{k}\right) - \frac{b_1xy_1}{(1 + fx + y_1 + gy_2)} - \frac{b_2xy_2}{1 + fx + y_1 + gy_2}$$
$$y_1{}' = -c_1y_1 - my_1y_2 + \frac{d_1xy_1}{(1 + fx + y_1 + gy_2)}$$
$$y_2{}' = -c_2y_2 - ny_1y_2 + \frac{d_2xy_2}{(1 + fx + y_1 + gy_2)}$$

Most new parameters are direct extensions of the previous model (b_2, c_2, d_2). The new term gy_2 in each denominator generalizes the ratio to consider all three species. New interaction terms between mountain lions and coyotes are also included (my_1y_2, ny_1y_2). Coyotes live in the same environments as mountain lions, and consume the same prey (deer). As mountain lions and coyotes are directly antagonistic to each other (beyond the indirect effects of consuming deer), both coefficients are negative. As mountain lions are far larger predators than coyotes (137 pounds vs 31 pounds), it can be assumed that n is the larger coefficient.

Third Model Behavior

A plot of all three populations is shown below. Using our starting conditions and parameters (as well as reasonable variations thereof), the mountain lions (green) and coyotes (blue) engage in conflict, suppressing the population of each. The mountain lions eventually win, sending the coyotes into extinction. A numeric solver was used to find the equilibrium that all three species approach.



Parameters:

$a = 0.875$, $b_1 = 1.8$, $b_2 = 1.8$,

$c_1 = 0.2$, $c_2 = 0.1$, $f = 0.44$,

$k = 1000$, $m = 0.007$, $n = 0.017$,

$d_1 = 0.1$, $d_2 = 0.092$, $g = 2$

Equilibrium: The deer population approaches 759.7, the mountain lion population approaches 44.5, and the coyote population approaches extinction.

Conclusion, Sources

While all three models share similar equilibrium points (between mountain lions and deer), behavior around these points differs substantially between models. Furthermore, each addition of complexity made our models more fragile. While both species refused to die in the first model, far more scenarios involved extinction in the third model. This could be due to the nature of natural ecosystems, or the nature of mathematical models.

Dubey, B., and Upadhyay, R.K. *Persistence and Extinction of One-Prey and Two-Predators System.* Nonlinear Analysis: Modeling and Control. 2004, Vol 9, No. 4.

Green et. al., *Reproductive Characteristics of Female White-Tailed Deer*, Theriogenology, Volume 94, 2017.

Longhurst et. al., *The California Deer Decline and Possibilities for Restoration*, California Nevada Wildlife Transactions, Wildlife Society, 1976.

Pettorelli et. al., *Predation, Individual Variability and Vertebrate Population Dynamics*, Oecologia, 2011.

Pierce, Becky et. al. *Selection of Mule Deer by Mountain Lions and Coyotes: Effects of Hunting Style, Body Size, and Reproductive Status."* 2000. Journal of Mammalogy, Volume 81, Issue 2.

Xiao, Dongmei and Ruan, Shigui. *Global dynamics of a ratio-dependent predator-prey system.* Journal of Mathematical Biology. 2000.

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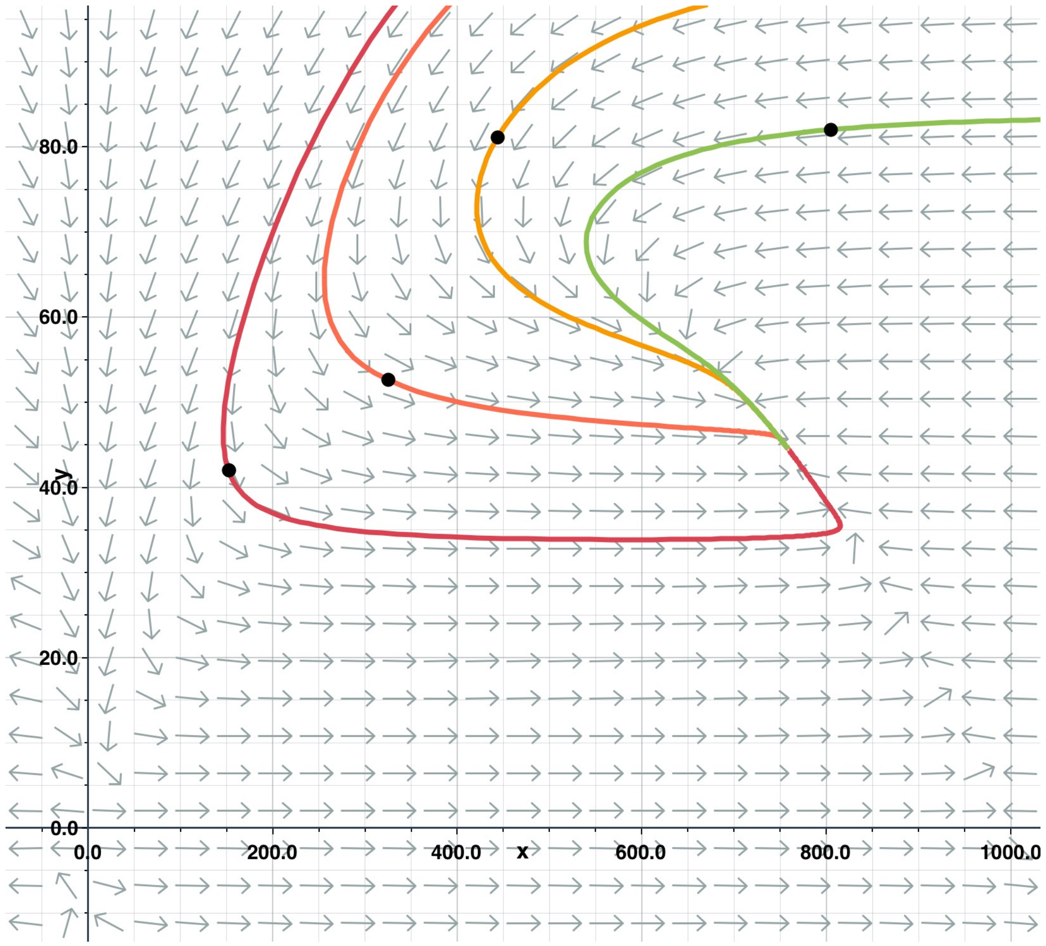

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Masked and Unmasked SIR Model

$$dS_M/dt = -a(1-b)^2 S_M I_M - a(1-b)S_M I_U$$

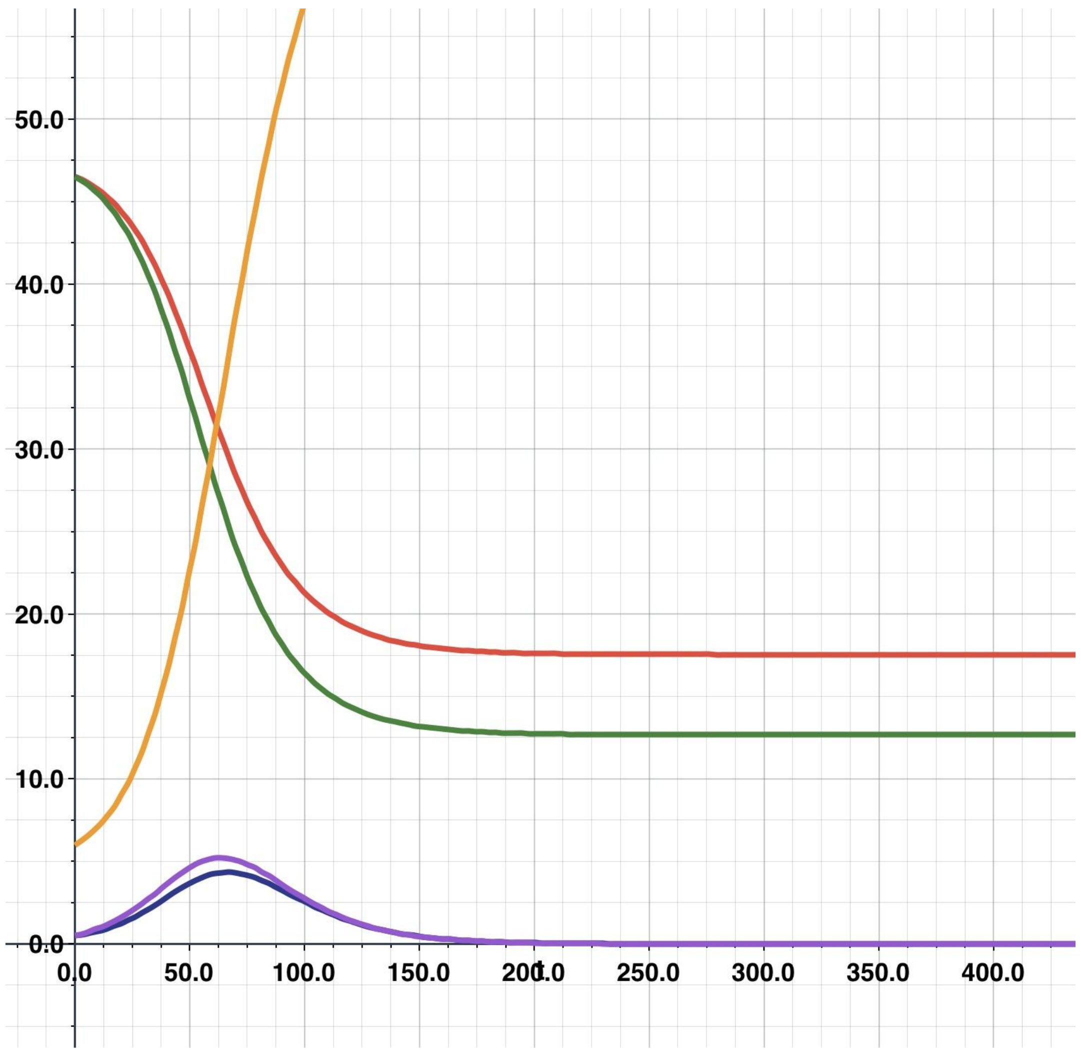
$$dS_U/dt = -a(1-b)S_U I_M - aS_U I_U$$

$$dI_M/dt = a(1-b)^2 S_M I_M + a(1-b)S_M I_U - cI_M$$

$$dI_U/dt = a(1-b)S_U I_M + aS_U I_U - cI_U$$

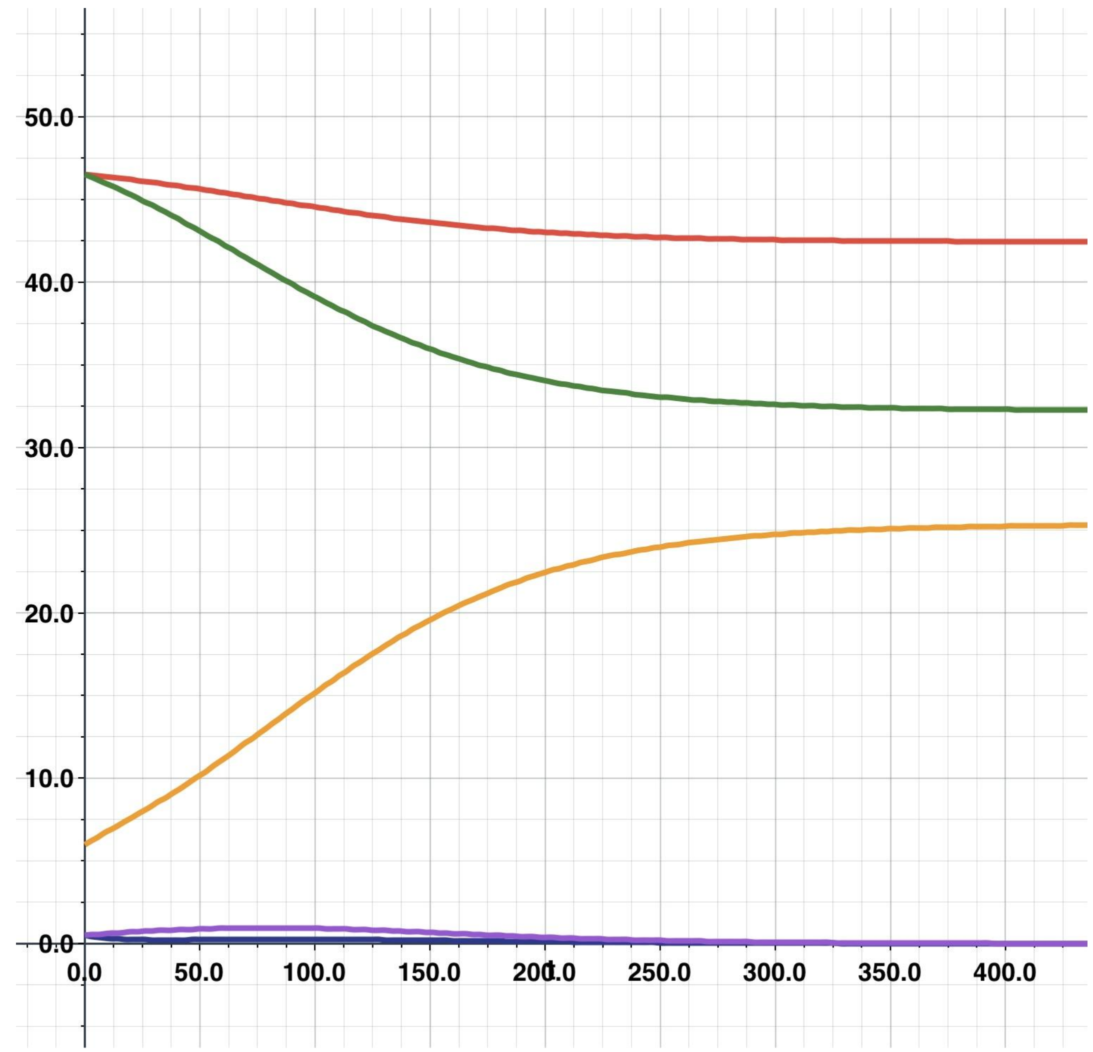
$$dR/dt = c(I_M + I_U)$$

Analysis of Varying Effectiveness



Time (days)

25% Mask Effectiveness



Time (days)

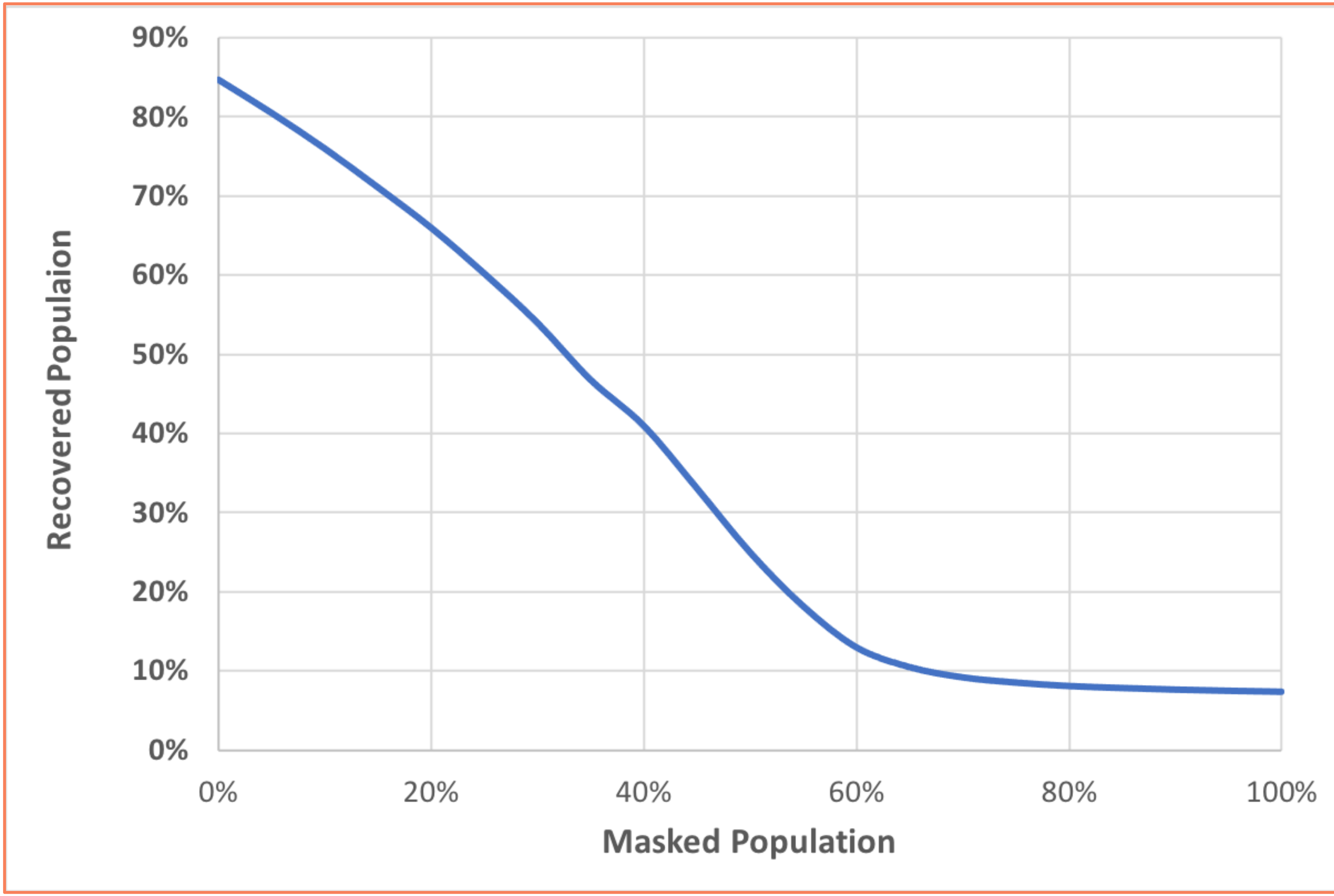
75% Mask Effectiveness

Red = S_M
Green = S_U
Blue = I_M
Purple = I_U
Yellow = R

Masked 50%
Unmasked 50%

Key observation: Wearing masks reduces infections in both the masked and unmasked populations.

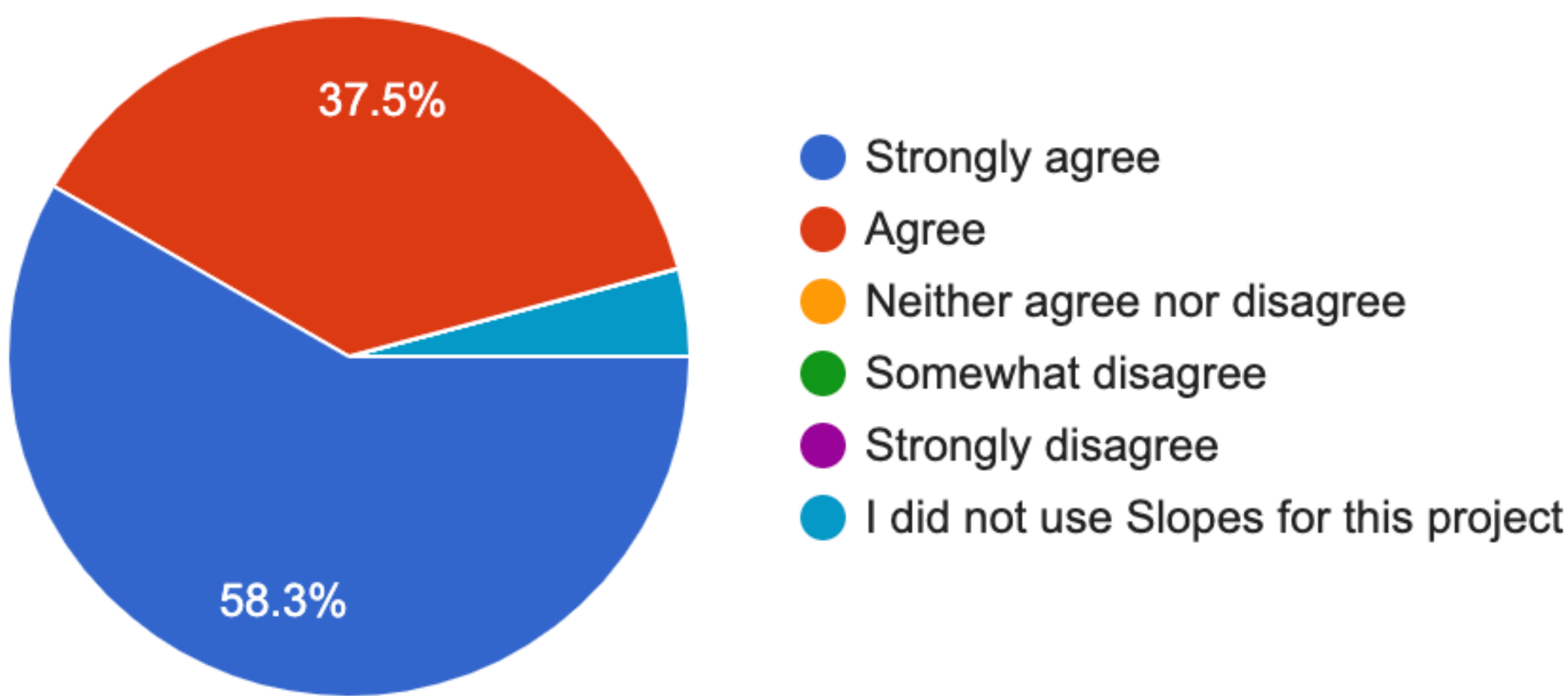
% Recovered vs % Masked Population

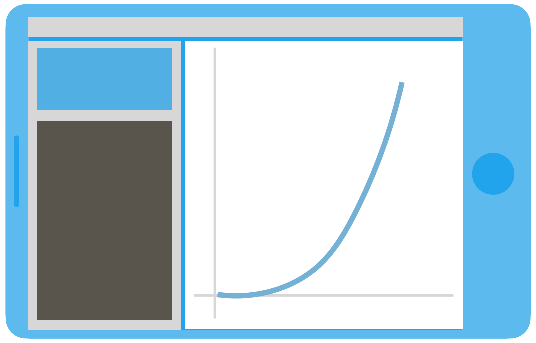


Mask effectiveness = 0.75

Student Feedback

To what extent do you agree with the following statement: "I feel that using Slopes increased my understanding of the mathematical models in my project."

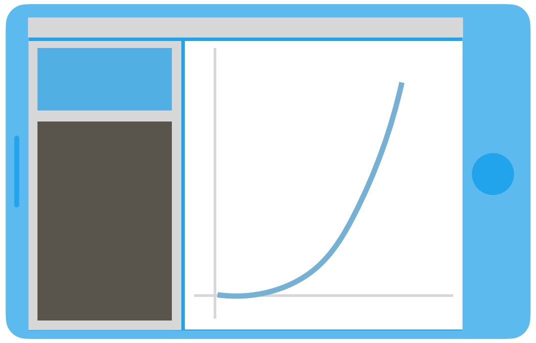




Student Feedback

"It's good just because visualizing helps a lot to be able to understand, especially when you get to higher levels of math and things get kind of hard to understand sometimes."

"I really love how interactive it is ...You can move it around and manipulate it. I like being able to click to see, okay, what does the solution with this initial condition look like?"



Student Learning

- Visualization and manipulation of mathematical models
- Engagement in mathematical conversation with peers and professor
- Demonstration of conceptual understanding (through activities and projects)

Using Slopes to Enhance Learning in Ordinary Differential Equations (K. Lucas and T. Lucas, 2022)

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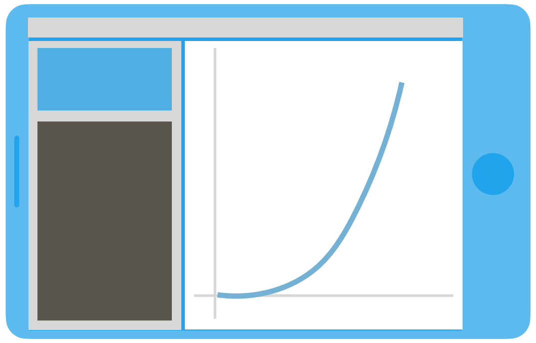

Questions?

Check out the apps at

http://www.slopesapp.com

http://www.wavespdeapp.com