Assessing Mathematical Virtues not just Skills

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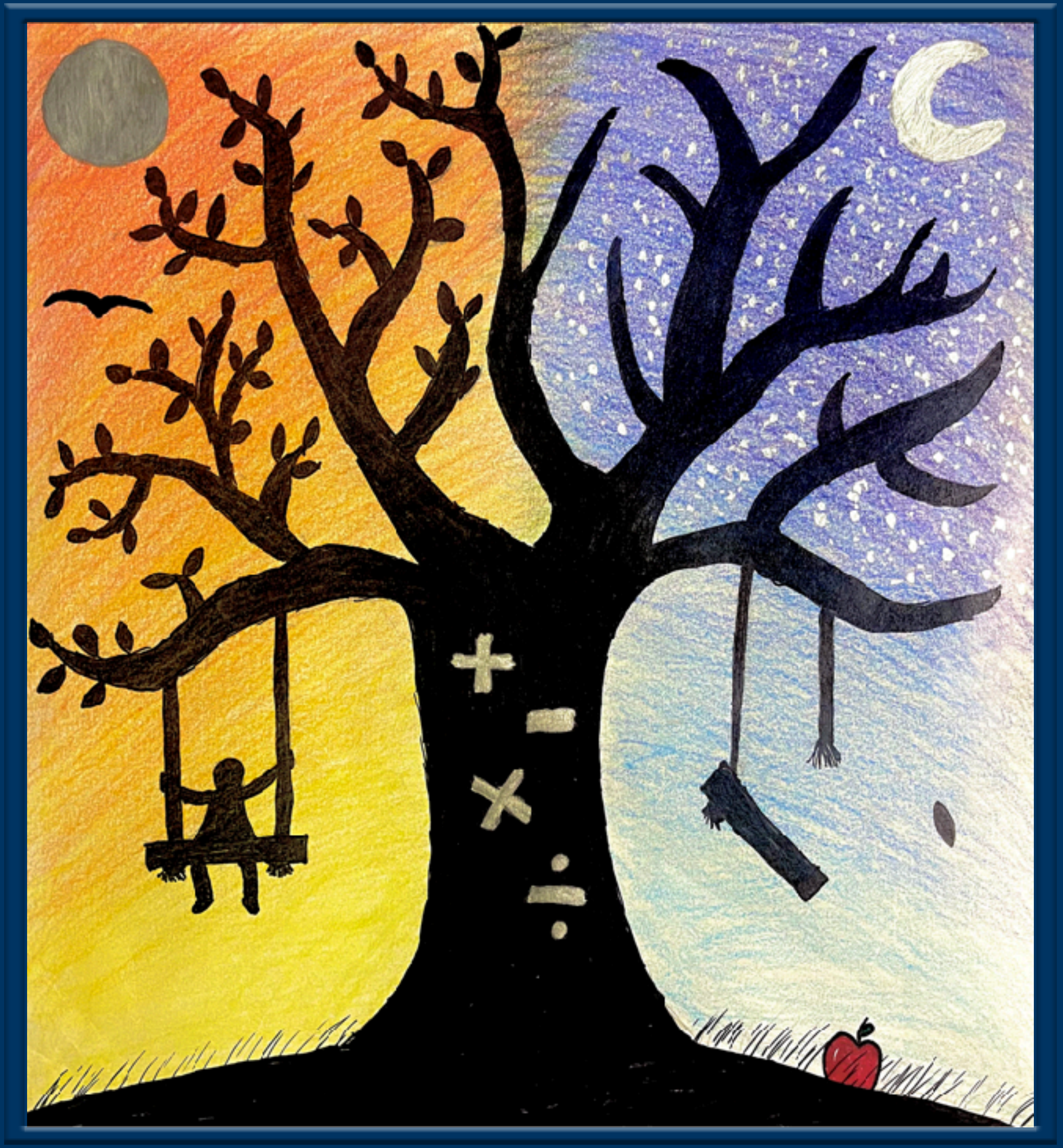

Mark Ying ‘25


HARVEY
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The Interview

- How many times is a toilet flushed in New York City during a commercial break of the Super Bowl?

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Jane Street

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Skills

- Number Facts
- Computing Things
- Factoring Polynomials
- Taking a Derivative

Virtues

- Persistence
- Creativity
- Abilities to interpret, define, quantify, abstract, visualize, strategize, model, generalize, collaborate
- Ability to Solve Problems You’ve Never Seen Before

Skills

- Number Facts
- Computing Things
- Factoring Polynomials
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replaceable

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what employers want

Like practice standards …

1. Making sense of problems and persevere in solving them,
2. Reasoning abstractly and quantitatively,
3. Constructing viable arguments and critiquing the reasoning of others,
4. Modeling with mathematics,
5. Using appropriate tools strategically,
6. Attending to precision,
7. Looking for and making use of structure, and
8. Looking for and expressing regularity in repeated reasoning.

- but more than standards.

Virtues offer more ways to see oneself as mathematical

- Math isn’t one-dimensional
- There are *many* mathematical virtues:

  playfulness, hopefulness, concentration, persistence, ability to change perspectives, disposition towards beauty, awe, joy, habits of generalization, interpretation, definition, quantification, abstraction, visualization, imagination, inventiveness, thinking for oneself, creation, structure identification, thirst for deep investigation, unflappability, seeing setbacks as springboards, hospitality, humility, teaching, mentoring, …

Virtue

- Excellence of character that leads to excellence of conduct
- *being* leads to *doing*
- *dispositions* or *habits of mind* lead to *practices*

Virtues - practically speaking

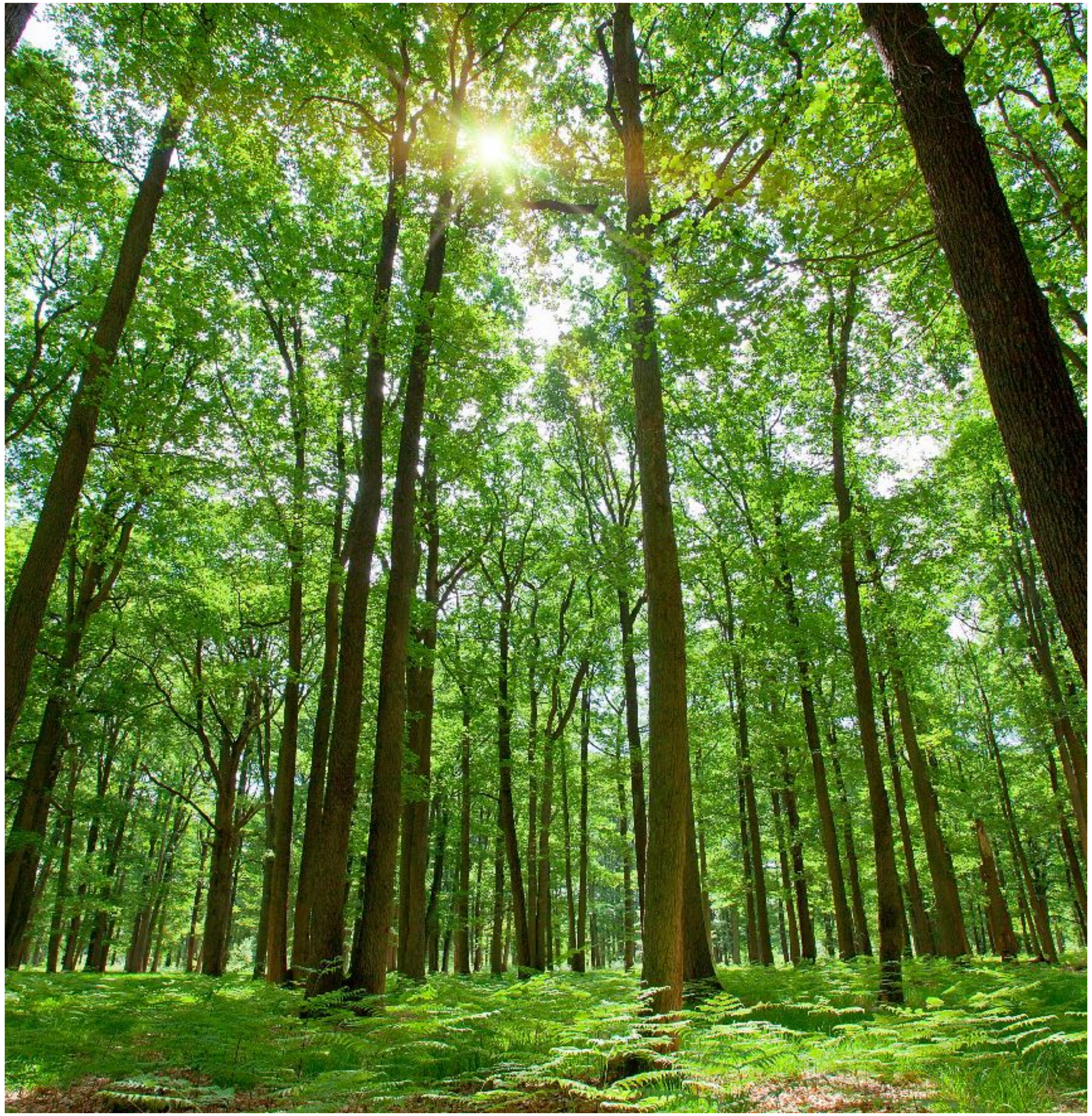
- Are what employers are looking for
- Can't be replaced by computers/AI
- Make your life richer no matter what you do

- Provide a better answer to the question

"Why do I need to know this stuff"?

Virtues are Human

- They’re what make the classroom, workplace, and your life experiences a place of wonder, delight, joy



Mudd Math

Goals

Mathematical Practice

P1. (Affection)

Students should appreciate the beauty, fun, and power of mathematics, and be able to articulate what mathematics is about and what mathematicians do.

P2. (Application)

Students should be able to link applications and theory, and be able to apply mathematics in a variety of settings.

P3. (Inquiry)

Students should develop mathematical independence and experience open-ended inquiry, so they have the competence and confidence to build on their knowledge base.

P4. (Communication)

Students should develop effective thinking and communication skills.

P5. (Technology)

Students should be able to use technological tools appropriately and effectively.

P6. (Society)

Students should strive to be good citizens who understand the impact of their work on society.

P7. (Teamwork)

Students should be able to function well as part of a team and have honed their leadership skills.

P8. (Diversity)

Students should be able to work and communicate with diverse groups of people of varying abilities who come from a variety of cultures.

https://www.hmc.edu/mathematics/department-of-mathematics-goals/

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- Expectation of Enchantment
- Hopefulness
- Disposition to Beauty
- Habits of Generalization
    - and more…

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virtues make your life richer

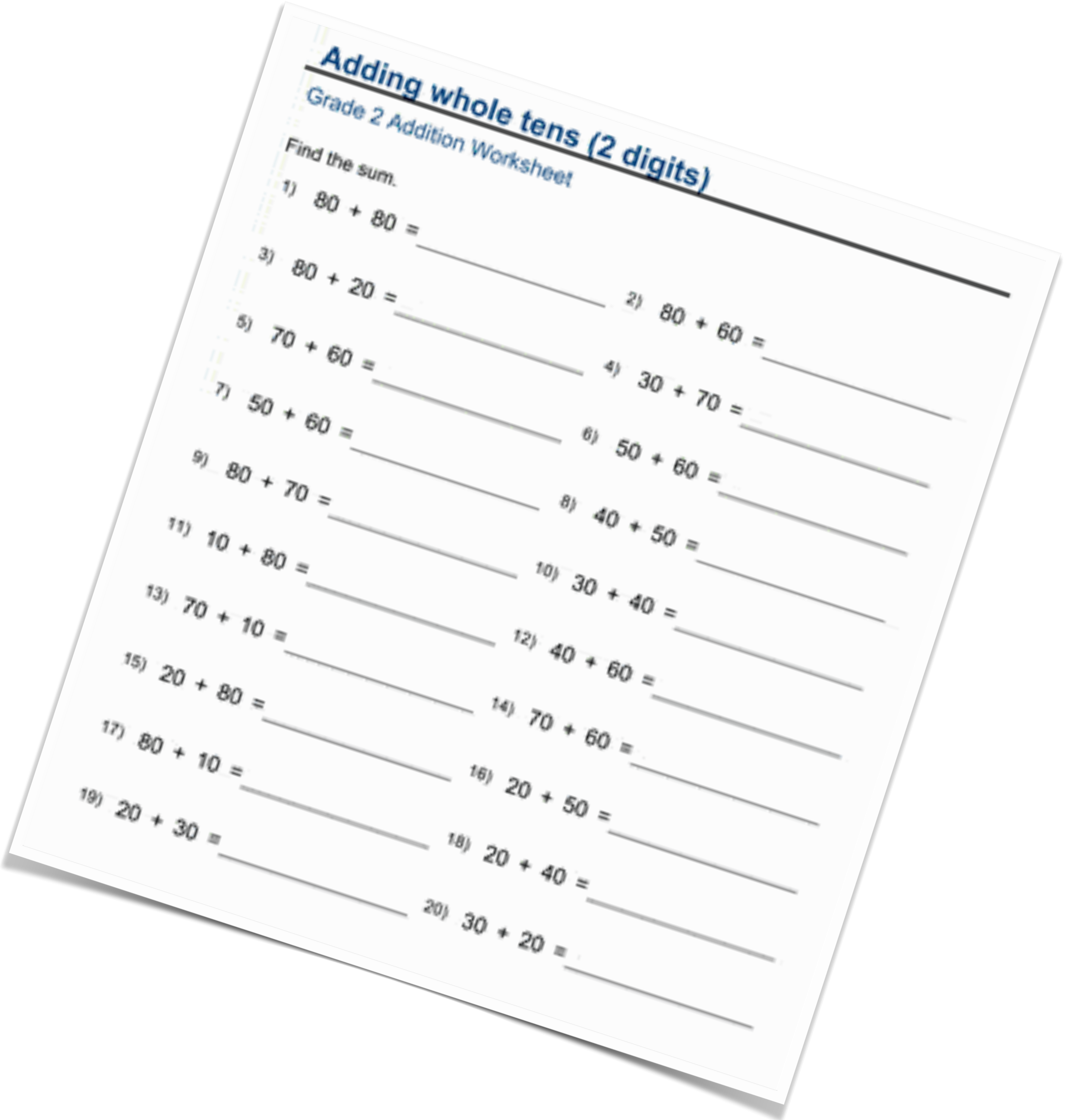
mathematics for human flourishing

What's Happened to Math Education?

- Why do we focus primarily on skills?
- Why do our homework and exam problems usually only explicitly measure skills?
- Why do we often strip all the best parts of doing mathematics—the parts that *make them feel human*—away from our classrooms?

The Problem:

Skills are easy to assess, virtues are not



Adding whole tens (2 digits)

Grade 2 Addition Worksheet

Find the sum.

1) 80 + 80 = ______

2) 80 + 60 = ______

3) 80 + 20 = ______

4) 30 + 70 = ______

5) 70 + 60 = ______

6) 50 + 60 = ______

7) 50 + 60 = ______

8) 40 + 50 = ______

9) 80 + 70 = ______

10) 30 + 40 = ______

11) 10 + 80 = ______

12) 40 + 60 = ______

13) 70 + 10 = ______

14) 70 + 60 = ______

15) 20 + 80 = ______

16) 20 + 50 = ______

17) 80 + 10 = ______

18) 20 + 40 = ______

19) 20 + 30 = ______

20) 30 + 20 = ______

- But that doesn't mean you shouldn't try!

Christopher Jackson

I've been studying in "Introduction to the Theory of Numbers" I'm only in chapter three though. Had a mishap with some of the notation kind of early but I figured it out (sometime a dot is multiplication sometimes it is an and, a +1 after a and/multiplication is a plus one and not part of the product), so it's alright. Only one thing bothers me about this book, it had no practice problems in it whatsoever. I don't know what kind of idea that is (ha-ha), it might be for a good reason though. But I work through the theorem with examples that I base on whatever the theorem is saying. From the little bit that I've seen it definitely seems like number theory is a beautiful subject.

- A traditional assessment would not reveal what he knows.

How do we assess virtues?

- I have only partial answers.
- But they've been effective in changing the culture of my classroom.
- Start somewhere!

“But this isn’t math.”

- YES, YES IT IS.
- Our job is to help our students see this! (Not just on assessments.)
- We can reframe for our students what math is.
  - Reinforce virtues as course goals.
    “One of the things I want you to learn is the value of struggle.”
  - Helps to have support of entire department.
    “A math department goal is clear communication.”

“But assessing virtues is subjective”

- EVERY ASSESSMENT IS SUBJECTIVE
    - the questions you choose to put on an exam are subjective
    - how many points you assign are subjective
    - rubrics can help you and students see what you’re after

Directly assessing virtue

- Example: Persistence
  - Do your assessments elicit evidence of persistence?
  - Do your assessments value it?
- Why not ask students to reflect on these virtues directly?
  - Low-stakes, formative metacognition good for students… and for us
  - Scaffold with clear instruction about what's expected (rubric)

Reflection Questions

Persistence:

Take one homework problem you have worked on this semester that you struggled to understand and solve, and explain how the struggle itself was valuable. In the context of this question, describe the struggle and how you overcame the struggle. You might also discuss whether struggling built aspects of character in you (e.g. endurance, self-confidence, competence to solve new problems) and how these virtues might benefit you in later ventures.

- See my blog post: “7 exam questions for a pandemic”

Reflection Questions

Strategization:

For any problems you cannot solve on this exam, suggest a strategy you might try to tackle the problem, and show what happened as a result. Describe any strategic gaps you were unable to bridge, and list 3 helpful insights that may help another person trying to tackle the problem. Doing so will earn you up to 1/2 credit on the problem.

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Assessing Reflection

- Don’t sweat it.
- Announce that you’ll give full credit for any thoughtful answer.
- It’s formative assessment.
- It communicates to students that virtues are just as valuable as skills.
- You’ll get to know your students as human beings.
- You’ll learn how your students think in ways that will benefit your teaching.
- You’ll enjoy reading them.

More Objections

"These are going to be difficult to grade."

Not if you make them low-stakes. Keep rubrics simple.

"Doesn't this reward students who can write better than others?"

Focus on ideas, not language skills.

"We'll never be able to perfectly assess these things."

Does that mean we shouldn't try? Are they better than our current practice?

"Won't these questions come as a surprise to students?"

Not if you make virtues part of your course goals & constantly reinforce them.

"I'm stressed and I can't add another thing to my plate."

I have a feeling, if you try this, you'll really *enjoy* reading the answers.

Reflection Questions

Curiosity:

What mathematical ideas are you curious to know more about as a result of taking this class? Give one example of a question about the material that you'd like to explore further, and describe why this is an interesting question to you.

- See my blog post: “7 exam questions for a pandemic”

Curiosity

- making connections

(10 pts) What mathematical ideas are you curious to know more about as a result of taking this class? Give an example of a question about the material that you'd like to explore further, and in a couple of paragraphs, elaborate on why this is an interesting question to you.

Throughout most of this class I keep getting the feeling that if I plunged my hand a little past the surface of Galois theory I would grasp hold of representation theory: We played with Galois groups that permuted the roots of polynomials and were literally subgroups of the symmetric group. We considered the elementary symmetric functions. We introduced the idea of a character for all of two seconds. All of this made me think if we used representation theory tools what more could we do?

Would the proofs be cleaner? Would representation theory open up new results for us? You just can't convince me with the hammer that representation theory is, and the mountains of symmetry that we played with in Galois theory, the disciplines to not talk to each-other.

The question I am most interested in asking is: What would the study of the irreducible representations of the Galois groups tell us about our splitting field, the polynomial, or the roots?

Curiosity

- making connections

I've taken this class while completing my senior thesis in game theory, and at first I thought that the two topics were fairly separate. I didn't immediately see the connections I could draw between the algebra and games I was studying, until I attended a seminar about Rubik's Cubes. Perhaps I wasn't understanding the sheer applicability of abstract algebra before attending, but the seminar gave me a taste of what algebra is capable of. I was already familiar with solving a Rubik's Cube, but the seminar approached the cube through the lens of group theory. Each of the possible turns you can perform on a cube can be thought of as operations or morphisms, and by combining turns together, you can create algorithms -- and subsequently more complex morphisms. You can define any state of the cube with a series of operations, and by doing so, you can answer a lot of questions like "what are illegal states?" and "how many possible legal states are there?" with some combinatorial arguments. This fusion of abstract algebra and combinatorics is something I've now been able to identify in our class, and with my own thesis topic.

When we discussed the straightedge-and-compass constructions in class, it took me a bit of studying to realize that this was incredibly similar to Rubik's Cube example and what I was trying to achieve in my own research. Using Galois theory, we were able to narrow down the possible pool of "moves" that we could take using a straightedge and compass down to moves of "order two or less". From there, we were able to definitively disprove some very classical problems such as squaring the circle and trisecting an angle, since these actions required the user to make "moves" that were of higher order. I've been thinking about them as "moves" because I've been trying to see how this type of thinking could extend into different games, in which people have restrictions in their moves and still must try to accomplish an end objective. With this thinking, I would like to be able to answer some of the "impossibility" questions present in other games.

Reflection Questions

Disposition toward beauty:

Consider one mathematical idea from the course that you have found beautiful, and explain why it is beautiful to you. Your answer should: (1) explain the idea in a way that could be understood by a classmate who has taken classes X and Y but has not yet taken this class and (2) address how this beauty is similar to or different from other kinds of beauty that human beings encounter.

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Disposition Towards Beauty

- “soulmates"

(b) This kind of beauty is reminiscent of the idea of a soul mate - someone who in some sense matches you perfectly and who you always want to be together with. Roots of the same minimal polynomial have the same "nature" in some way, and never occur without each other. The idea of two separate things being connected on some deep level also occurs in a lot ideas of dualism (you can't have shadow without light, you can't have cold without warmth, etc.) which have been present in religions and human beliefs for longer than recorded history, and must appeal to us on some deep level. I guess it is beautiful to see that this idea that is so deeply ingrained in us has an equivalent which can be proved even in the world of mathematics.

Disposition Towards Beauty

- “traditional vs. modern art”

I find the depth of constructions similar to the relationship between traditional and modern art. The construction of the equilateral triangle is easily appreciated, just like a traditional painting. Both are accessible. On the other hand, the construction of a 17-gon is difficult to understand, and ever more difficult to attempt to replicate. When constructions are taken to their limits, they are obtuse. I feel like modern art is in a similar position. In my art history class last semester, my classmates and I were often left puzzled after Prof. Fandell discussed a banana taped to a wall or a toilet made of gold. We often had to ask, “what is the meaning of this piece?” Someone seeing the construction of a 17-gon for first time might ask the same question. Ultimately, I believe that the accessibility and universal nature of these constructions make them beautiful.

Disposition Towards Beauty

- “everybody can find their own splitting field”

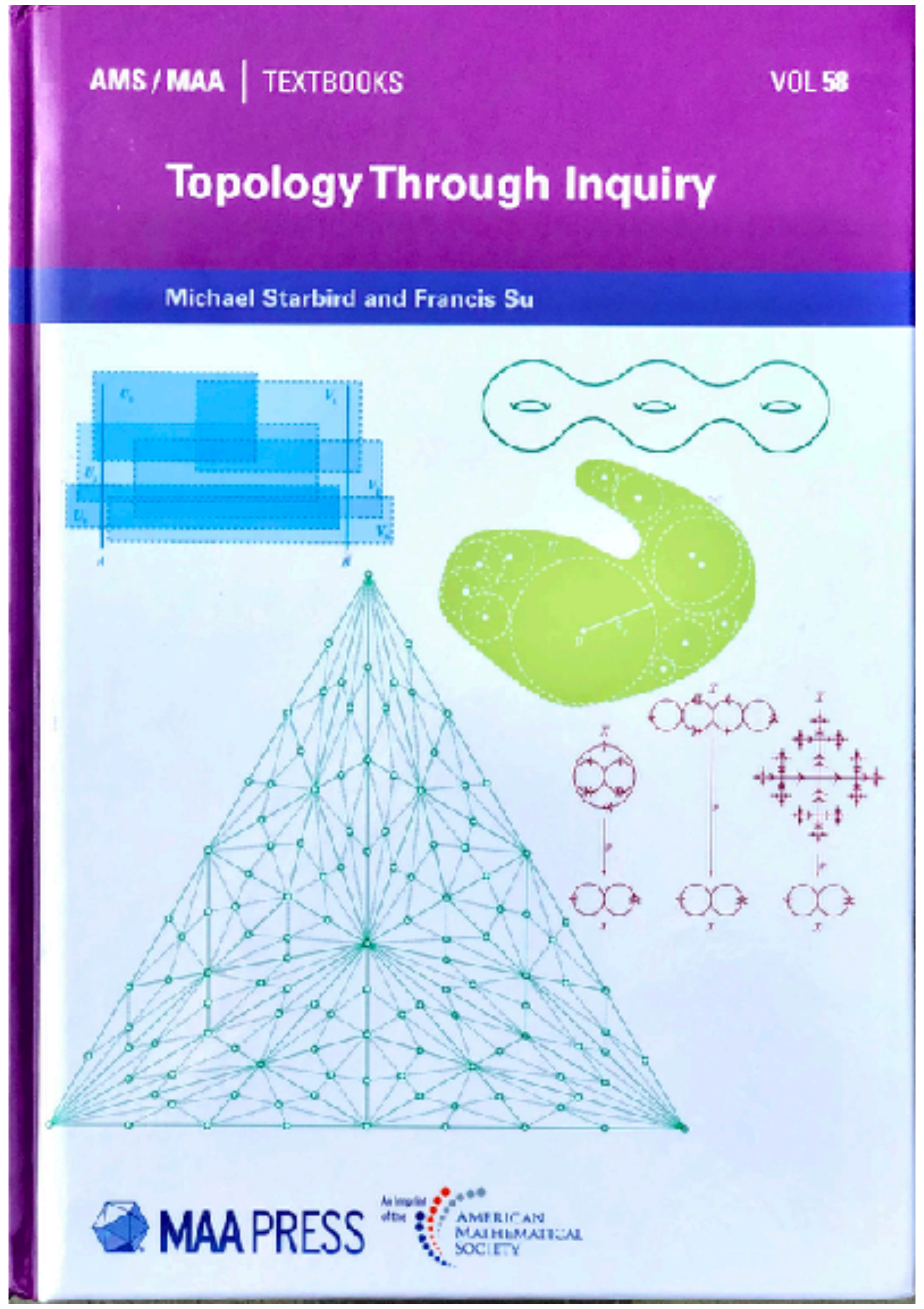
I think what I find beautiful about splitting fields can be explained through something I find really beautiful about Pomona. There’s a lot that I don’t like about Pomona, but something that I will concede is that Pomona has taken strides to have a diverse student body. Pomona has students with wildly different identities and backgrounds, and there are actually a fair amount of student-run organizations around campus which support minorities and underrepresented identities. I’m sure that there is still work to be done, but this concept is beautiful in my eyes: the ability to have a space in which every individual can find their own metaphorical splitting field, where their strange roots can be actually recognized. Apologies if this analogy is a bit of a stretch, but in my mind, a lot of people have “roots” in some sense which are not fully recognized by society, and I think people are starting to recognize this. Effectively, my generation grew up in a world of rationals, and we’re just now beginning to discover that we’ve had complex roots all along. Queer roots, low-income roots, underrepresented ethnic identity roots, and Pomona has provided me with a place where I’ve been able to find a splitting field for me. This, for me, is part of the beauty of Pomona, and I identify that beauty

What you'll learn

How did the ideas of this course enlarge your sense of what it means to do mathematics?

One student responded: *This class gave me a much better understanding of what it means to do mathematics than I had in the past. Most of our problem sets in other classes were applying theorems that we learned in class, and the problems were roughly of comparable difficulty. However, with this class, we did much of the learning on our own, through results that we proved. In addition, some of the problems were relatively straightforward, but there were several very challenging problems, where my group didn't even have a clear idea where to start. This seems much more realistic to the life of a mathematician, where problems don't present themselves in homogeneous sets.*

Effective Thinking Principles



Effective Thinking Principle. *Weaken Hypotheses if Possible.* To understand theorems better and to improve them if possible, identify exactly what aspects of the hypotheses were actually used in the proof.

One of the early theorems you proved about compactness concerned limit points. You proved earlier that in a compact space, every infinite set has a limit point. Actually, only the countable compactness property was needed to draw that conclusion. In fact, the issue of convergence basically characterizes countably compact spaces.

Effective Thinking Principle. *Make Connections.* When you observe similarities in apparently different contexts, ask if there is a reason.

When we started investigating methods for recognizing holes in spaces, our first attempt led to the idea of the fundamental group. That idea involved looking at loops that surround holes and viewing two loops as equivalent if they were homotopic. If we think of a loop as a map of a circle, then a homotopy is the map of a cylinder. So

Effective Thinking Principle. *Pin Down Intuition.* Take the trouble to pin down details that justify intuition. Either you will better understand why your intuition is correct or you will realize you are mistaken—both good outcomes.

Q. Identify 10 Effective Thinking Principles you've incorporated into your practice of mathematics. Describe specifically how 3 of them have helped you solve problems you've encountered.

Next Step: Rubrics

Heidi Wall

Phillips Academy

Learning Goal	6- Exemplary	5- Demonstrating	4-Progressing	3- Emerging	0-2 Not Meeting Expectations
Reasoning and Proof	I use evidence to justify statements or make conclusions or in critiquing arguments posed by others. My arguments are clear, concise, and demonstrate certain efficiencies by utilizing already proved theorems.	I can formulate an argument based on logic and truths. My reasoning may not be as efficient as it could be.	My solution begins correctly, but I am missing some steps that may lead to an incorrect answer or incorrect justification.	I am not clear on which strategy to use. My solutions lack supporting evidence and may be off topic. I have little work to support my thinking or that work may show a lack of connections.	I am unclear on methods or off topic. My solutions are incorrect or non-existent. I have little to no work to support my thinking.
Modeling and Representation	I always apply appropriate models to represent and understand quantitative and geometric relationships.	I mostly apply appropriate models to represent and understand quantitative and geometric relationships.	I show some understanding of applying a model to represent a relationship but may be missing some connections.	I may try and apply a studied model to the wrong scenario.	I have a hard time recognizing that a model may help to represent a given relationship.
Problem Solving	I am able to understand what problems are asking and develop multiple solution pathways. This includes problems that require applying understanding to new situations. My solutions are accurate, with special attention to detail. I also verify that my answers make sense independently and in context.	I am able to understand what the problem is asking and can develop a solution on my own. I sometimes can apply this understanding to new situations. I can solve problems with an alternative approach if prompted, but I usually have a preference for one method. If asked, I could verify my answer. My solutions are correct. Any small errors do not affect the overall demonstration of understanding.	I am able to model problems after similar ones by replicating the process we did in class. I know one way to solve problems. My solution may not be the most efficient and I am not sure if I am correct.	I have trouble identifying what problems are asking without prompts from others. I need to work on my independence and recognizing clues, patterns, and the big picture.	I struggle to understand what a question is asking even with prompts from others.
Connections	I can see the connection between the problem and the bigger picture of our studies.	I mostly see the connections between problems and the bigger picture of our studies.	I may not see the connection between problems and others and don't always apply my understanding to new situations.	I don't always see the connection between problems and other concepts or topics.	I have trouble seeing connections.
Communication	My solutions are clearly communicated using consistent and correct mathematical notation. The solution is neat and easy to follow.	My work is easy to follow and I have used pictures, numbers and words to explain my thinking.	My work may be a little hard to follow, but I could explain it if prompted.	My work is confusing and hard to follow. I am unable to communicate a clear solution both orally and in writing.	My work is difficult to follow or non-existent.

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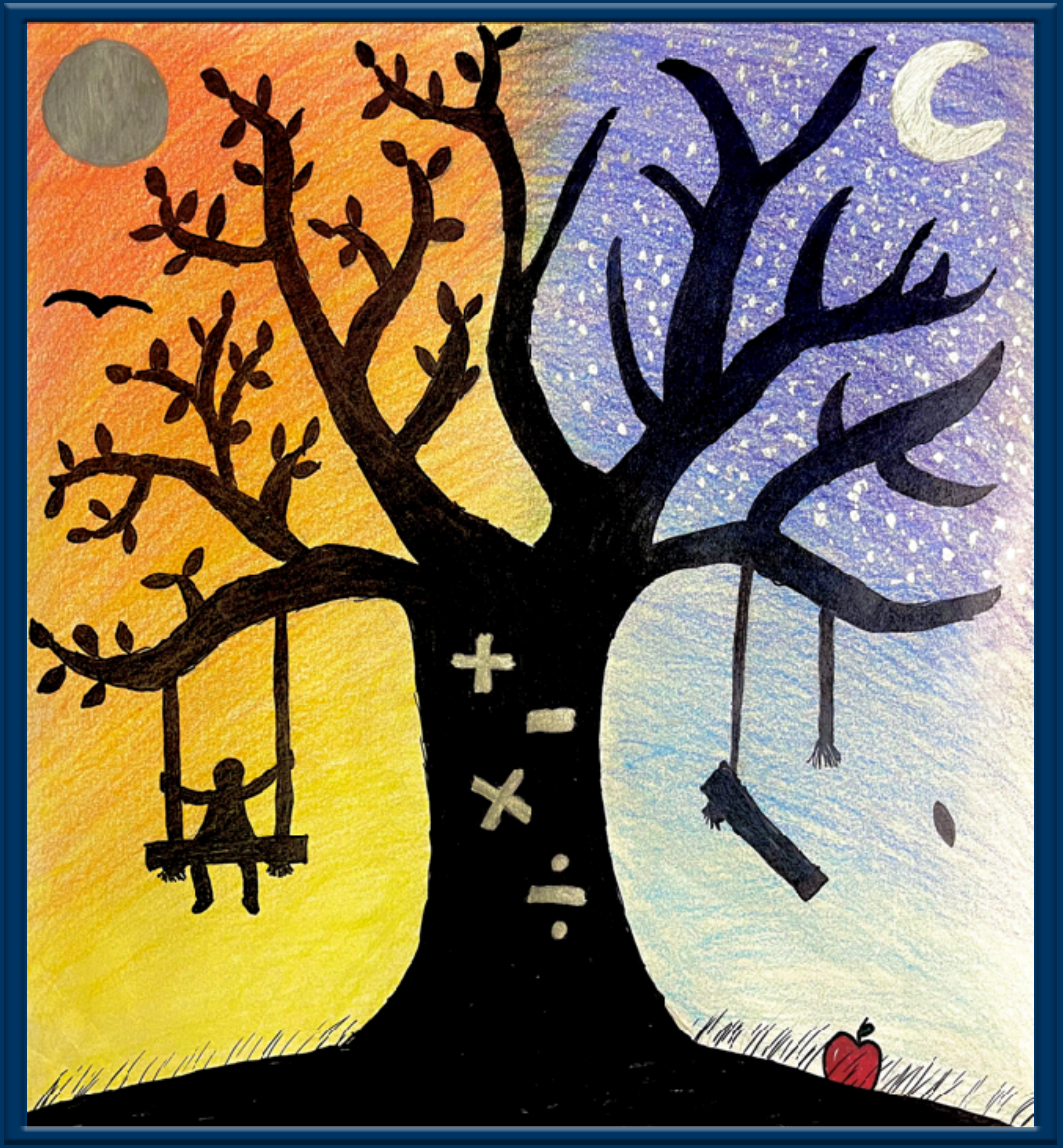

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