The Inquiry-Oriented Linear Algebra Project

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OLSUME

Online Seminar On Undergraduate Mathematics Education

IOLA: What is it?

The Inquiry-Oriented Linear Algebra (IOLA) project develops research-based student materials composed of challenging and coherent task sequences that facilitate an inquiry-oriented approach to the teaching and learning of linear algebra. The project also develops instructional support materials to help instructors implement the IOLA tasks in their classrooms.

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http://iola.math.vt.edu

How do we think about Inquiry?

- In terms of what students do and what instructors do in relation to student activity: (Rasmussen & Kwon, 2007)
  - Students learn mathematics through inquiry as they work on challenging problems that engage them in authentic mathematical practices
  - Instructors engage in inquiry by listening to student ideas, responding to student thinking, and using student thinking to advance the mathematical agenda of the classroom community

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- Four research-based goals in IBL classrooms: (Rasmussen, Marrongelle, Kwon, & Hodge, 2017)
  - Get students to share their thinking
  - Help students to orient to and engage in others' thinking
  - Help students deepen their thinking
  - Build on and extend student ideas

- Four key components of inquiry-oriented instruction: (Kuster, Johnson, Keene, & Andrews-Larson, 2018)
  - Generating student reasoning
  - Building on student reasoning
  - Developing a shared understanding
  - Formalizing language and notation

How do we think about Inquiry?

The role of the teacher is to: (Kuster, Johnson, Rupnow, & Garrison, 2019)

1. Facilitate student engagement in meaningful tasks and mathematical activity related to an important mathematical point
2. Elicit student reasoning and contributions
3. Actively inquire into student thinking
4. Be responsive to student contributions and use student contributions to inform the lesson
5. Engage students in one another's reasoning
6. Guide and manage the development of the mathematical agenda
7. Support formalization of student ideas and introduce language and notation when appropriate

Inquiry-Oriented Instruction

- Defining characteristics:
  - The central role of Realistic Mathematics Education as an instructional design theory
  - The curriculum materials go through some sort of iterative design, trial, and refinement cycle

- Research programs foundational to the IOI movement at the university level within the US:
  - Inquiry-Oriented Differential Equations (IODE; Rasmussen et al., 2007; 2018)

    iode.wordpress.ncsu.edu

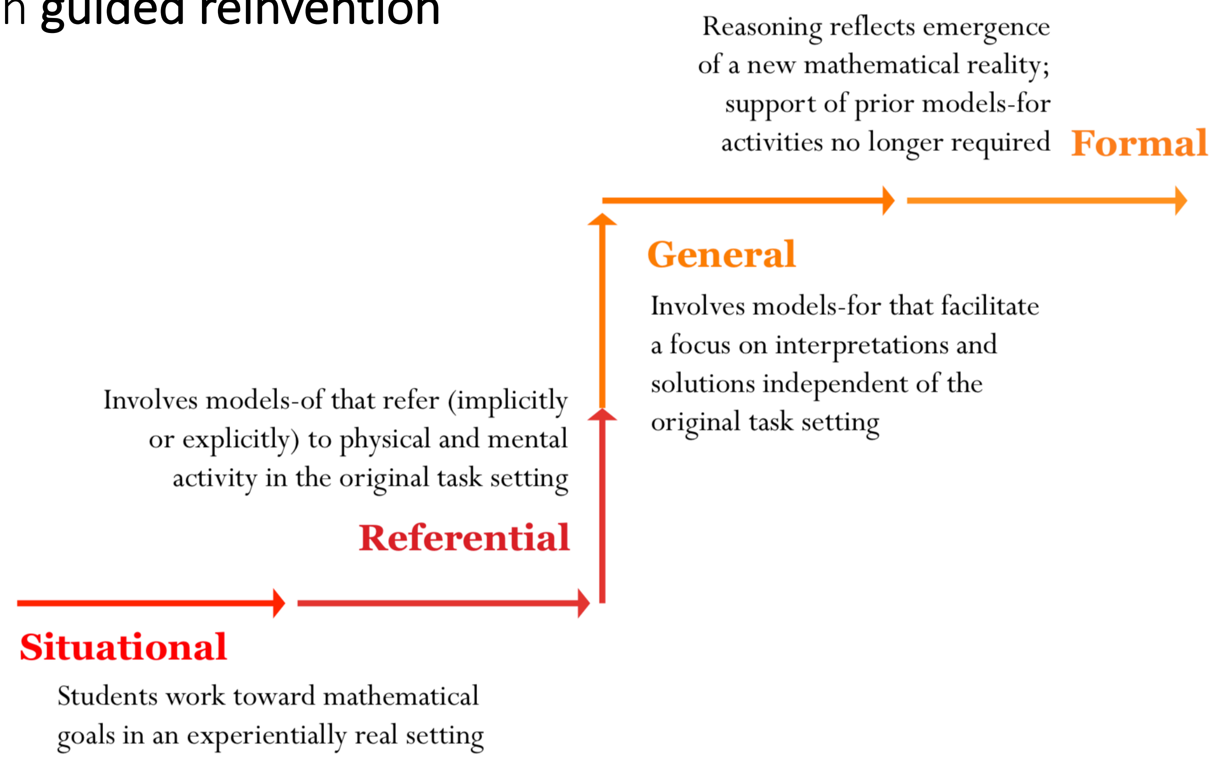
  - Inquiry-Oriented Abstract Algebra (IOAA, Larsen et al., 2013)

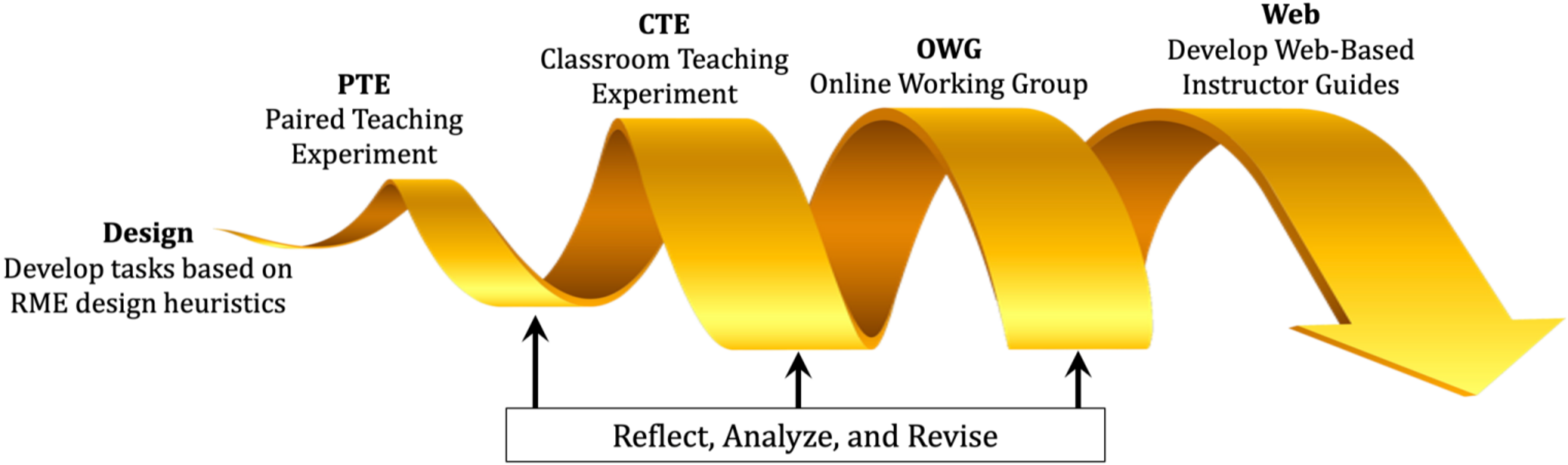
    taafu.org

Realistic Mathematics Education

- **Instructional design theory** originating in the Netherlands (Freudenthal, 1991; Gravemeijer, 1999) built on the tenet of mathematics as a human activity
- Start with students' current ways of reasoning to build toward more formal, sophisticated mathematics through **guided reinvention**



Design Research Spiral



Wawro, Andrews-Larson, Plaxco, & Zandieh (in press)

Context: Introductory Linear Algebra in the US

- Population is most likely:
  - Second-year students or first-year students with Advanced Placement (AP) credit
  - Students majoring in Engineering, Computer Science, Mathematics, Physics, Statistics, Economics
- Course most likely:
  - Is required for the aforementioned students
  - Has prerequisite courses such as Calculus I and sometimes Calculus II
  - Focuses on topics in $\mathbb{R}^n$
  - Is not rigorously proof-based
  - Topics are vast and vary (see next slide)

Introductory Linear Algebra topics

Table 1. Topics appropriate for a first course.

- Systems of linear equations.
- Properties of $\mathbf{R}^n$. Linear independence, span, bases, and dimension.
- Matrix algebra. Column space, row space, null space.
- Linear maps. Matrices of a linear map with respect to bases; the advantages of a change of basis that leads to a simplified matrix and simplified description of a linear map.
- Matrix multiplication and composition of linear maps, with motivation and applications.
- Invertible matrices and invertible linear maps.
- Eigenvalues and eigenvectors.
- Determinant of a matrix as the area/volume scaling factor of the linear map described by the matrix.
- The dot product in $\mathbf{R}^n$. Orthogonality, orthonormal bases, Gram-Schmidt process, least squares.
- Symmetric matrices and orthogonal diagonalization. Singular value decomposition.
- Orthogonal and positive definite matrices.

The Linear Algebra Curriculum Study Group (LACSG 2.0) Recommendations

Sepideh Stewart, Sheldon Axler, Robert Beezer, Eugene Boman, Minerva Catral, Guershon Harel, Judith McDonald, David Strong, and Megan Wawro

Universally covered topics	1. Solving systems using row reduction (97%) 2. Eigenvectors/values (97%) 3. Determining linear independence/dependence of a set (94%) 4. Dot products (92%) 5. Characteristic polynomials (92%) 6. Diagonalization (92%) 7. Determinant formulas (91%) 8. Producing bases for subspaces (91%)
Often covered topics	1. Fundamental subspaces of a matrix (83%) 2. Similar matrices (77%) 3. Geometric / algebraic multiplicity (72%) 4. Non-diagonalizable matrices (69%) 5. Gram-Schmidt orthogonalization (64%) 6. Function spaces / polynomial spaces (63%) 7. Change of basis (58%)
Sometimes covered topics	1. Determinants as volumes (52%) 2. Least squares (50%) 3. Complex numbers (48%) 4. Abstract vector spaces (42%) 5. Inner product spaces (36%)

Figure 1. Percent of respondents who report covering each topic.

Report on a US-Canadian Faculty Survey on Undergraduate Linear Algebra

Could Linear Algebra Be an Alternate First Collegiate Math Course?

Christine Andrews-Larson, Jason Siefken, and Rahul Simha

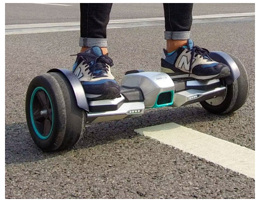
Unit 1: Magic Carpet Ride Sequence

- Was created to support students' understanding of linear combinations, span, and linear independence
- Begins the course with a focus on vectors, their algebraic and geometric representations, and build towards their properties as sets
- Starting with this instructional sequence fosters:
    - The initiation of formal ways of reasoning about vectors as the 'objects' of linear algebra to be investigated and understood
    - A coordinated perspective between algebraic and geometric views of vectors and vector equations
    - An intellectual need (Harel, 2007) for sophisticated solution techniques and strategies

Wawro, Rasmussen, Zandieh, Sweeney, & Larson (2012)

The Setting for Unit 1: Travel

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:



This Photo by Unknown Author is licensed under CC BY-SA-NC

We denote the restriction on the *hover board's* movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

This Photo by Unknown Author is licensed under CC BY-NC-ND

We denote the restriction on the *magic carpet's* movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

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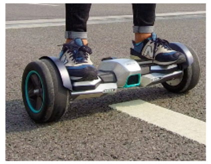
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Task 1: Getting to Gauss

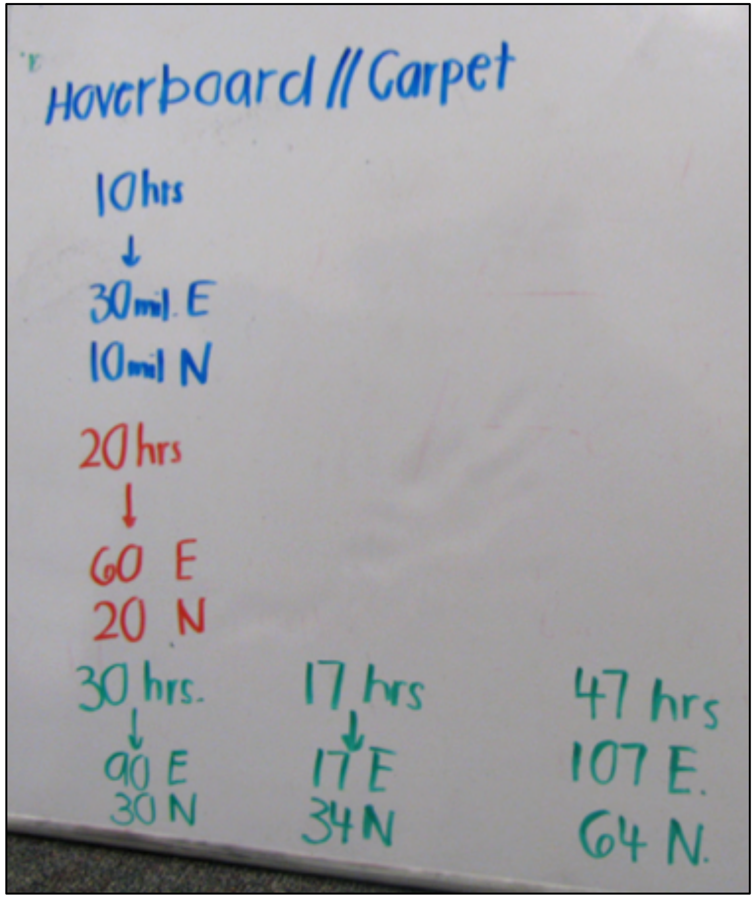
Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss.
Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

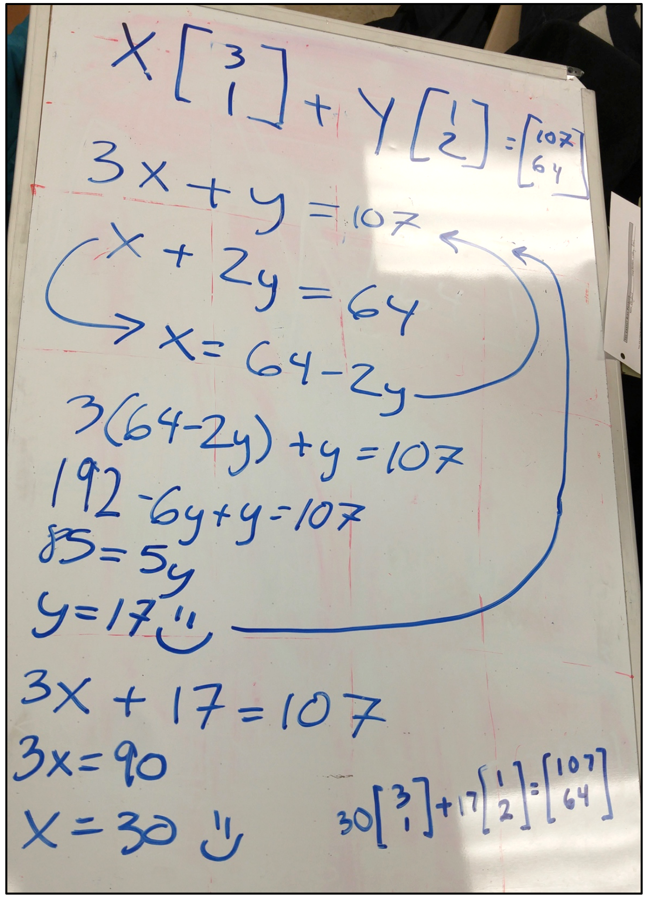
Your goal:

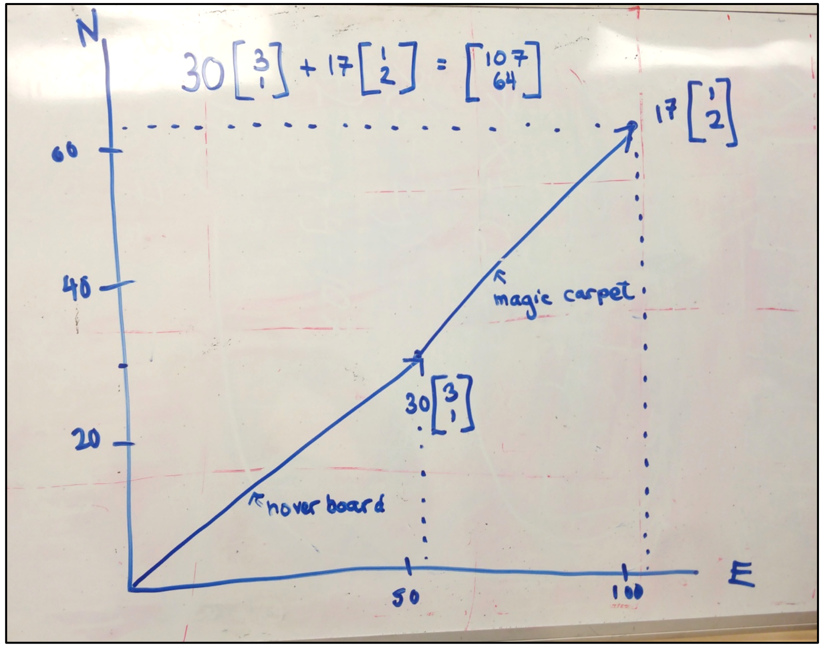
Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin. If so, how?
If it is not possible to get to the cabin with these modes of transportation, why is that the case?

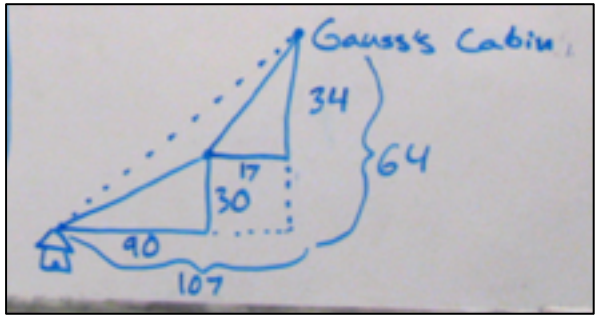
As a group, state and explain your answer(s) on your group's whiteboard. Use the vector notation for each mode of transportation as part of your explanation and use a diagram or graphic to help illustrate your point(s).

Task 1: Examples of Student Work









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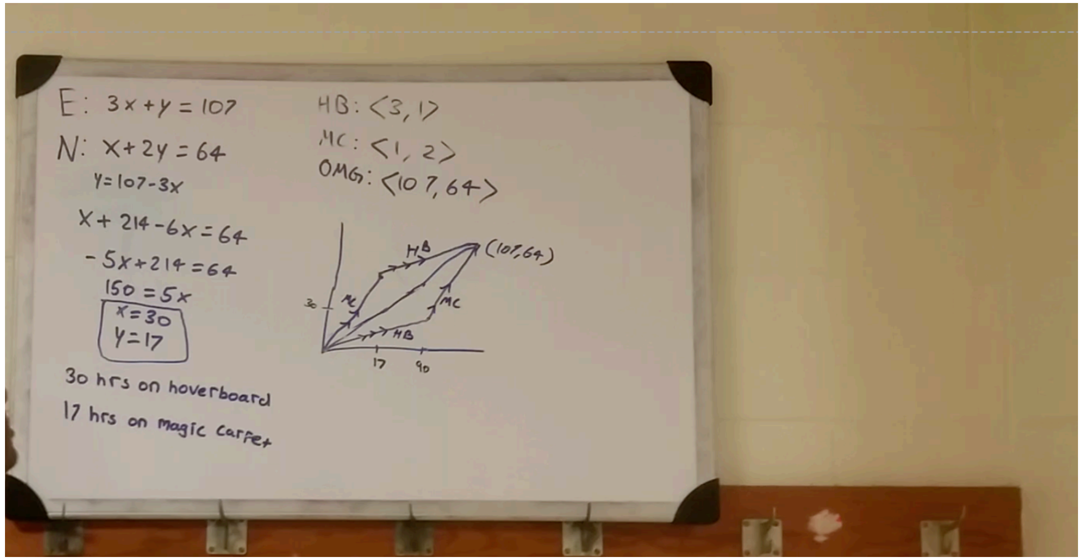
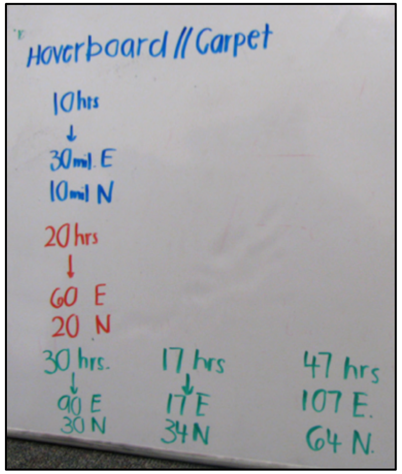
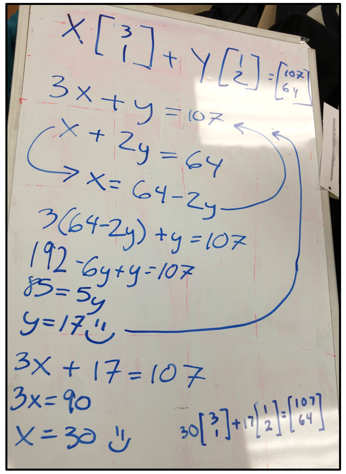
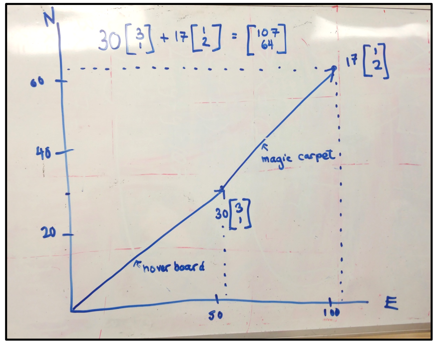


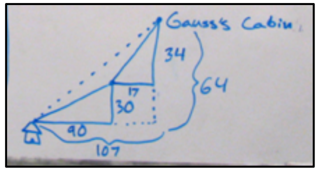
"We set up a system of equations, and we split up the x and y compo-, or the east and north components, rather. Uh, you have the hoverboard, which the basic unit vector was 3,1. And the magic carpet was 1,2. And you had to get to 107,64. <u>Obviously none of these could go into 107,64 just by themselves</u>, so we had to use some combination of the magic carpet and the hoverboard. So we did was we did 3 times x that would give us – this was for the east component – that would give us 3x and that represented the hoverboard. And plus y, and that represents the magic carpet. And for the north we did x + 2y equals 64 because that represents the north components for the hoverboard and magic carpet. And when we solved for that we got x = 30 and y = 17. So you'd have to spent 30 hours on the hoverboard and 17 hours on the magic carpet, <u>but it really doesn't matter what order you go in</u>. As you can see here, if you go the hoverboard first and then you go the magic carpet second. So, but here you can go the magic carpet first and the hoverboard second and still reach the same destination."

Task 1: Outcomes from the task

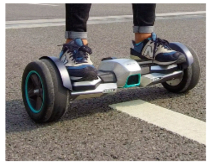








- Students are introduced to the expectation to work in groups, discuss, and present mathematical thinking
- Students hear and appreciate multiple solution strategies
- Teacher tags student work and introduces formal notation for scalar multiplication, linear combinations, vector equation, system of equations, and solution
- The class begins to coordinate geometric and algebraic views of linear combinations of vectors



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Task 2: Can Gauss Hide?

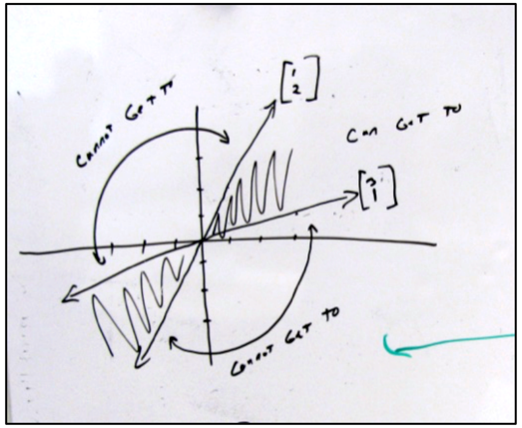
Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him

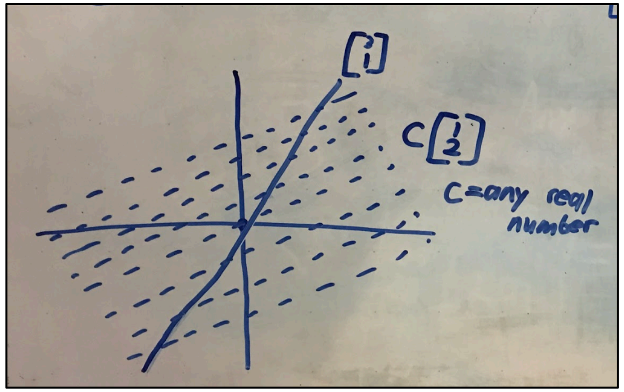
Your Goal:

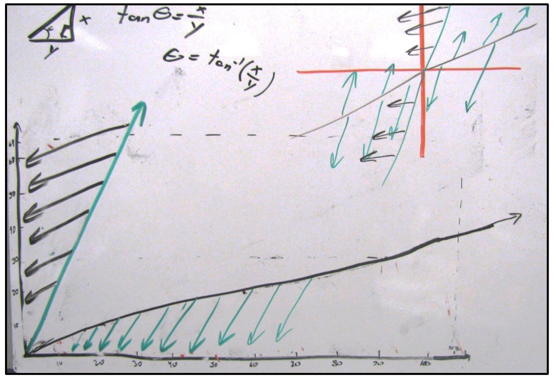
Are there some locations that he can hide and you cannot reach him with these two modes of transportation?
Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these graphically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

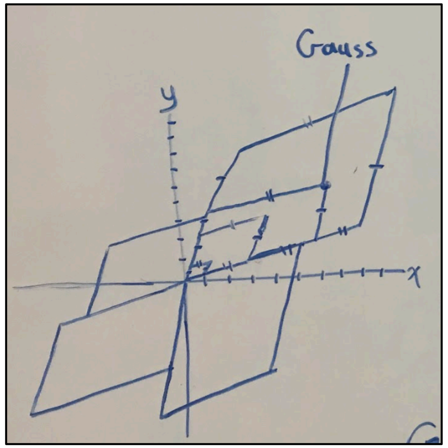
Use your group's Jamboard slide as a space to collaborate and work on this problem and communicate your thoughts.

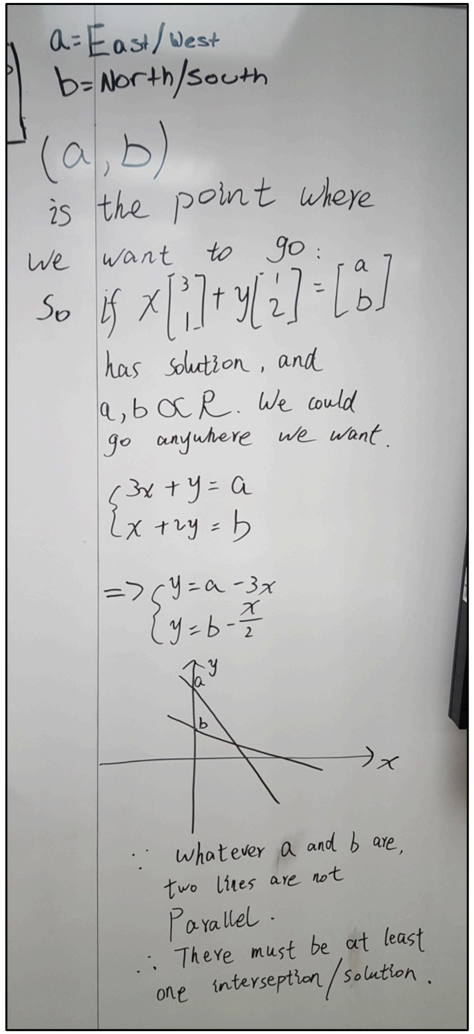
Task 2: Examples of Student Work

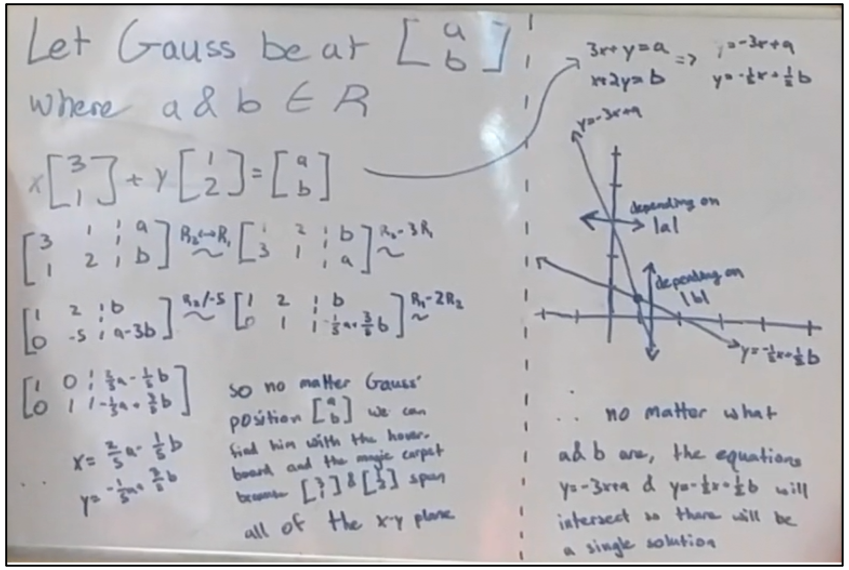












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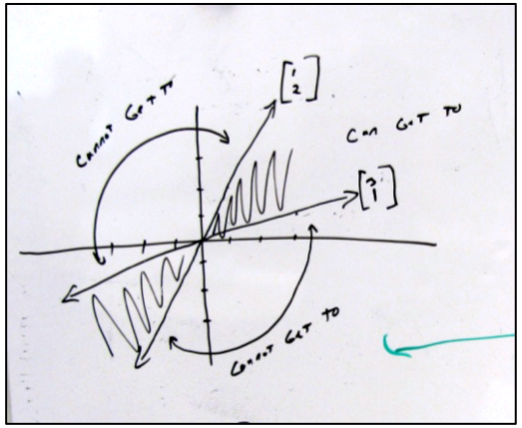

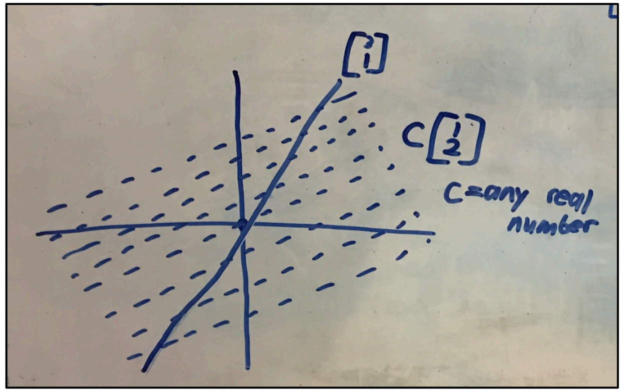

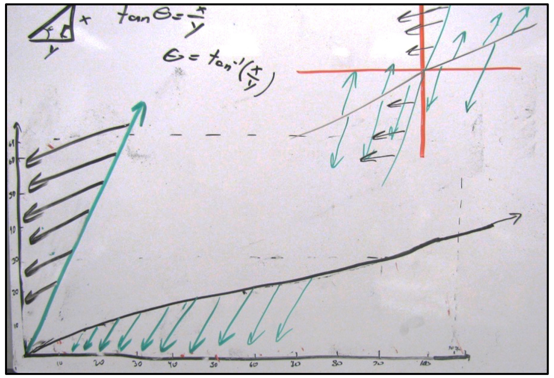

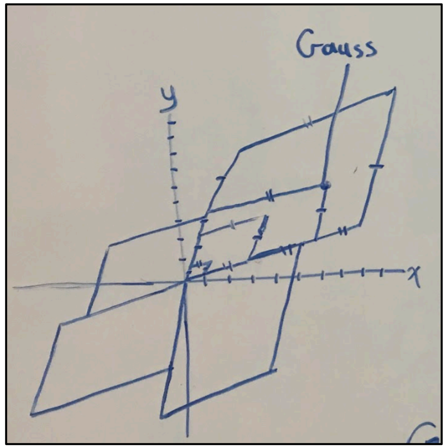

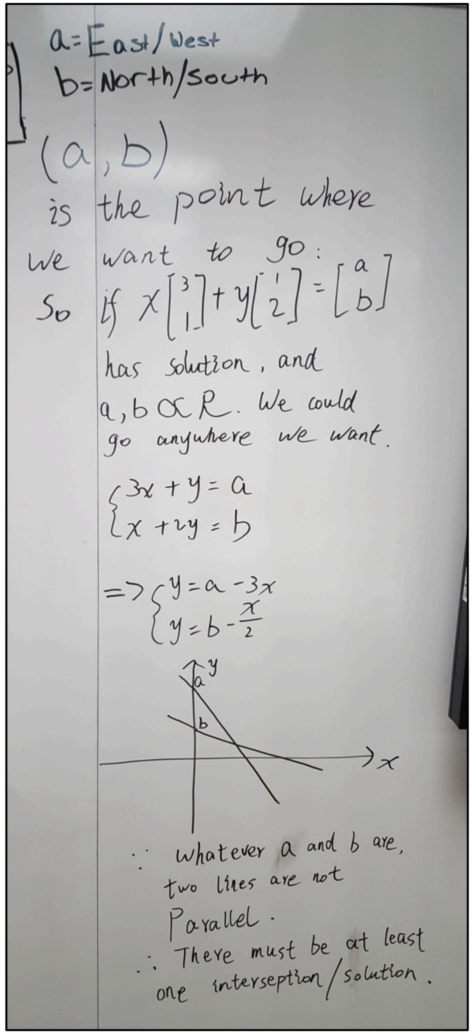

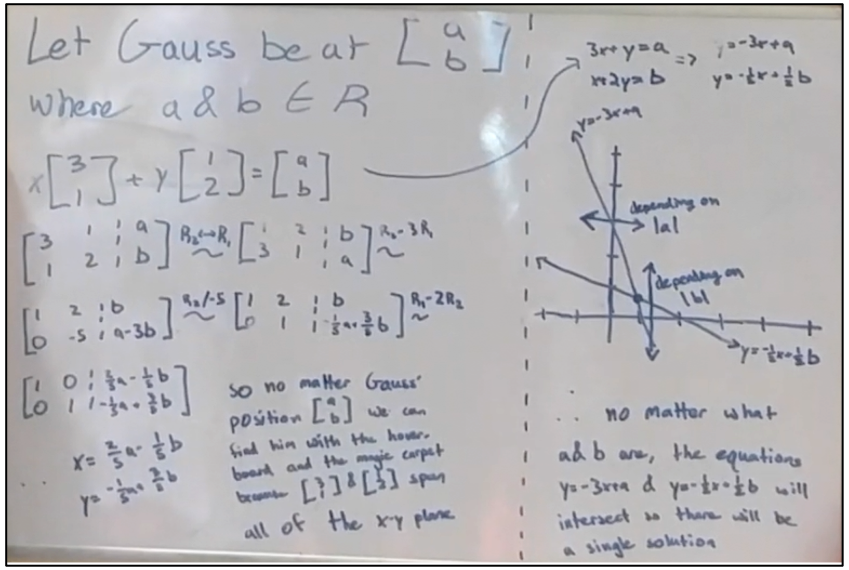

Task 2: formalizing student work with new terms

The *span* of a set:

- The **span** of a set of vectors is all of the places in a space you could get to with those vectors
- The **span** of a set of vectors is all possible linear combinations of those vectors
- If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k$ is called the **span** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k$ and is denoted by $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$

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Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$. When is a vector $\boldsymbol{v}$ in the ***span*** of S?

A vector $\boldsymbol{v}$ is in the span of S, written $\boldsymbol{v} \in span(S)$, if:

- You can travel to $\boldsymbol{v}$ using the vectors in S.
- There exists scalars $c_1, c_2, \ldots, ck$ such that $\boldsymbol{v}$ can be written as a linear combination of the vectors of S.
- There exists scalars $c_1, c_2, \ldots, ck$ such that such that $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$.

Task 2: Referring back to task setting to try new problems

Determine the following. You can express your answer in words, symbols, and/or a graph.

1. $\text{Span}\left\{\begin{bmatrix}3\\1\end{bmatrix}\right\}$
2. $\text{Span}\left\{\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}4\\6\end{bmatrix}\right\}$
3. $\text{Span}\left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}4\\5\end{bmatrix}\right\}$
4. $\text{Span}\left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\-6\end{bmatrix}, \begin{bmatrix}4\\5\end{bmatrix}\right\}$
5. $\text{Span}\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right\}$
6. $\text{Span}\left\{\begin{bmatrix}-3\\0\\-2\end{bmatrix}\right\}$
7. Is $\begin{bmatrix}107\\64\end{bmatrix}$ in the span of $\left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$? Why or why not?
8. Is $\begin{bmatrix}1\\0\end{bmatrix}$ in the span of $\left\{\begin{bmatrix}-2\\3\end{bmatrix}, \begin{bmatrix}4\\-1\end{bmatrix}\right\}$? Why or why not?
9. Is $\begin{bmatrix}3\\4\\-2\end{bmatrix}$ in the span of $\left\{\begin{bmatrix}-3\\0\\-2\end{bmatrix}\right\}$? Why or why not?

What conjectures do you have about what might be true with respect to span?

Task 3: Getting Back Home

Suppose you are now in a three-dimensional world for the travel problem, and you have three modes of transportation:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} \quad \boldsymbol{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

You are only allowed to use each mode of transportation once (in the forward or backward direction) for a fixed amount of time (c_1 on $\boldsymbol{v}_1$, c_2 on $\boldsymbol{v}_2$, c_3 on $\boldsymbol{v}_3$).

Find the amounts of time on each mode of transportation (c_1, c_2, and c_3, respectively) needed to go on a journey that starts and ends at <u>home</u> OR explain why it is not possible to do so.

Use your group's Jamboard slide as a space to collaborate and work on this problem. Use graphs, computations, mathematical symbols, etc. as you need to communicate your thoughts.

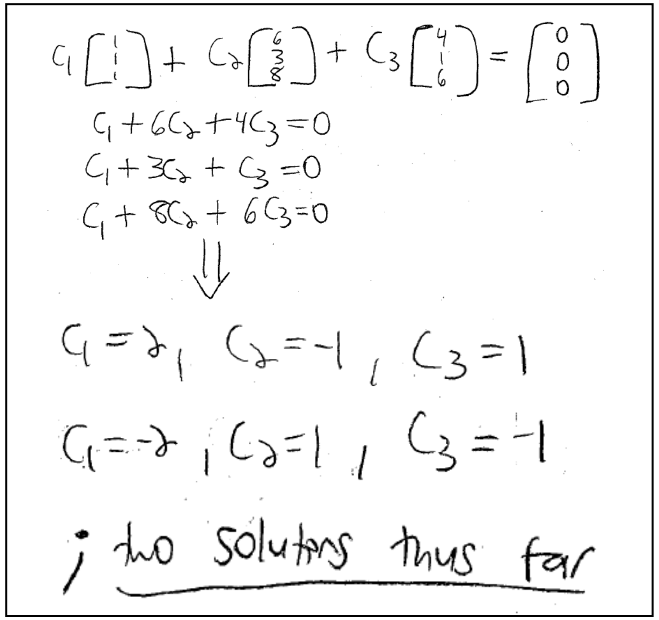
Task 3 follow-up questions

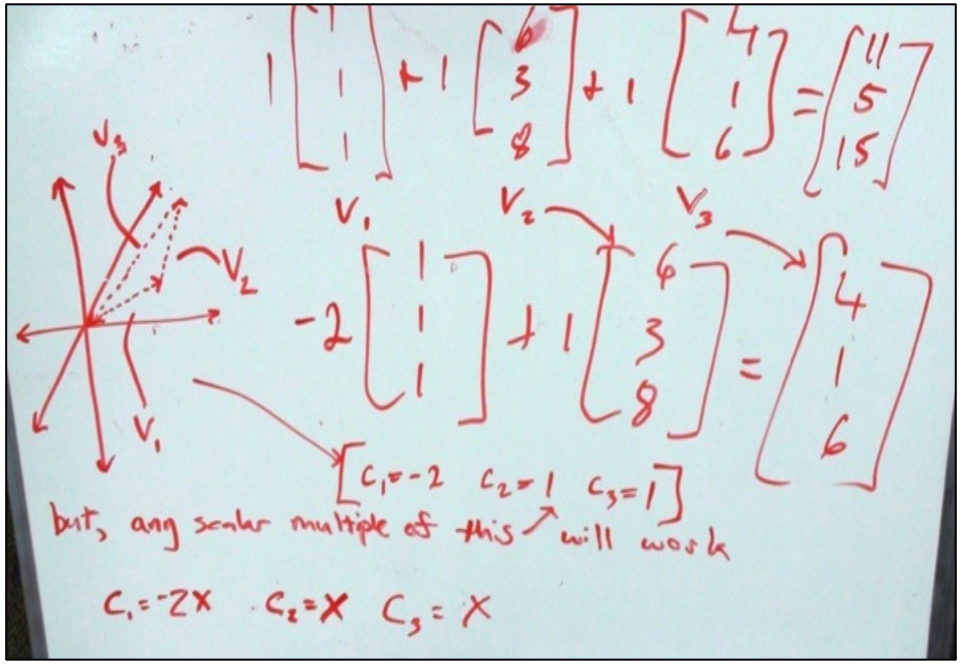
1. Is there more than one way to make a journey that meets the requirements of Task 3? (In other words, is there more than one solution to the relevant vector equation?) If so, what?

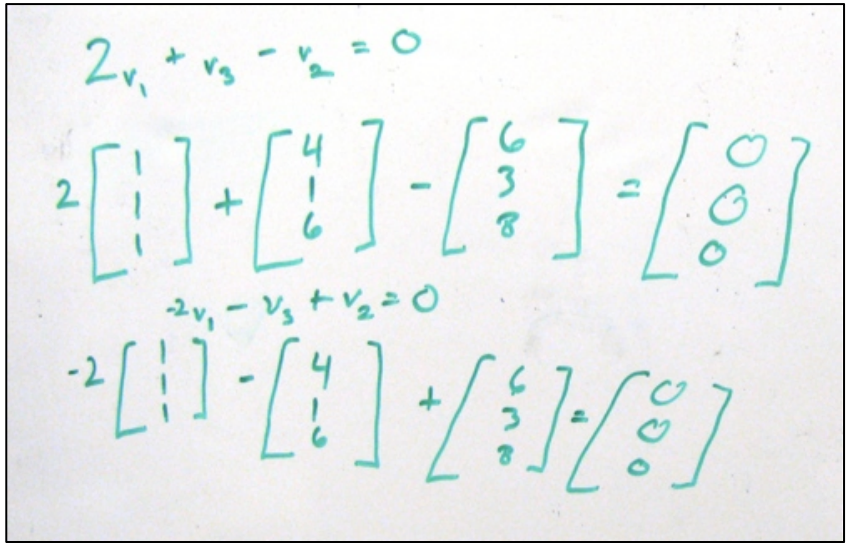
2. Is there anywhere in this 3D world that Gauss could hide from you? If so, where? If not, why not?

3. What is $span\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}6\\3\\8\end{bmatrix}, \begin{bmatrix}4\\1\\6\end{bmatrix}\right\}$?

Task 3: Examples of Student Work







Task 3: formalizing student work with new terms

Linearly Dependent Set

- A set of vectors is called **linearly dependent** if you can make a nontrivial trip that begins and ends at home
- A set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$ is called **linearly dependent** if there are scalars $c_1, c_2, \dots, c_k$, at least one of which is not zero, such that $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \mathbf{0}$
  - In this case, we'd say there is a nontrivial solution to the equation $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \mathbf{0}$.

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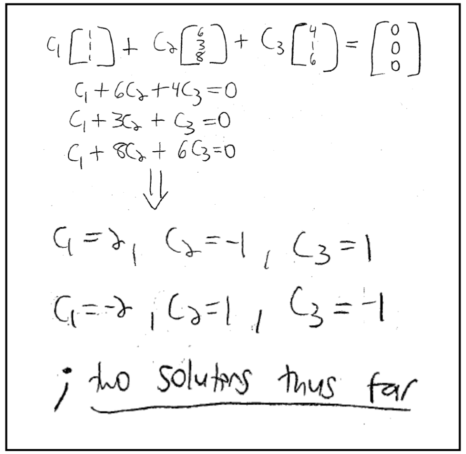
Linearly Dependent Set

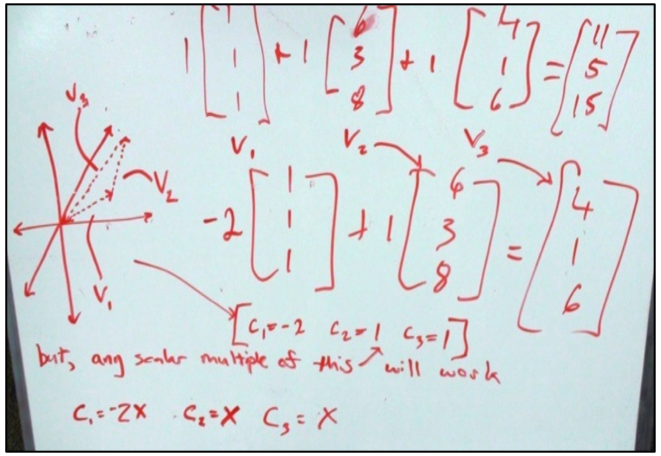
- A set of vectors is called **linearly dependent** if you can make a nontrivial trip that begins and ends at home
- A set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$ is called **linearly dependent** if there are scalars $c_1, c_2, \dots, c_k$, at least one of which is not zero, such that $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$
  - In this case, we'd say there is a nontrivial solution to the equation $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$.

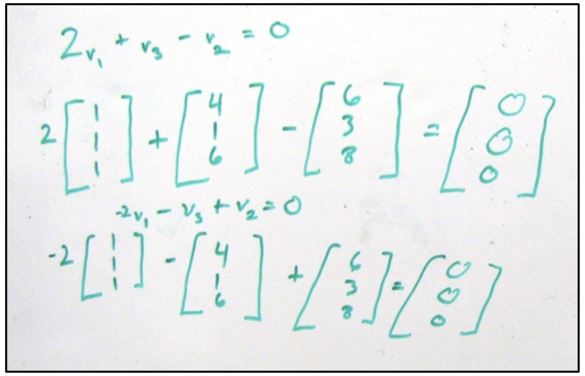
Linearly Independent Set

- A set of vectors is called **linearly independent** if you can't make a nontrivial trip that begins and ends at home
- A set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$ is **linearly independent** if the only solution to the equation $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$ is when all scalars $c_1, c_2, \dots, c_k$ are zero.
  - In this case, we'd say the only solution to the equation $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$ is the trivial solution.
- A set of vectors that is not linearly dependent is called **linearly independent.**

Task 3: Outcomes from the task







- Students investigate existence and uniqueness of solutions to homogeneous vector equations by considering “journeys that begin and end at home”
- Teacher tags student work and introduces formal notation for linearly dependent or linearly independent sets Continue to work with linear independence and dependence with other examples
- Connect back to the ideas of reaching Gauss (solutions to vector equations) and span but in 3-space

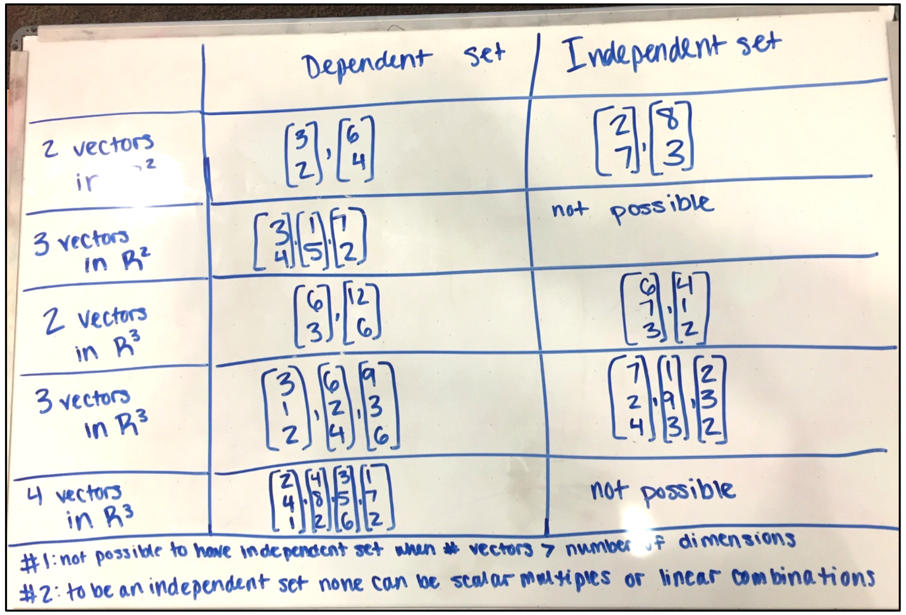
Task 4: Creating Examples & Generalizations

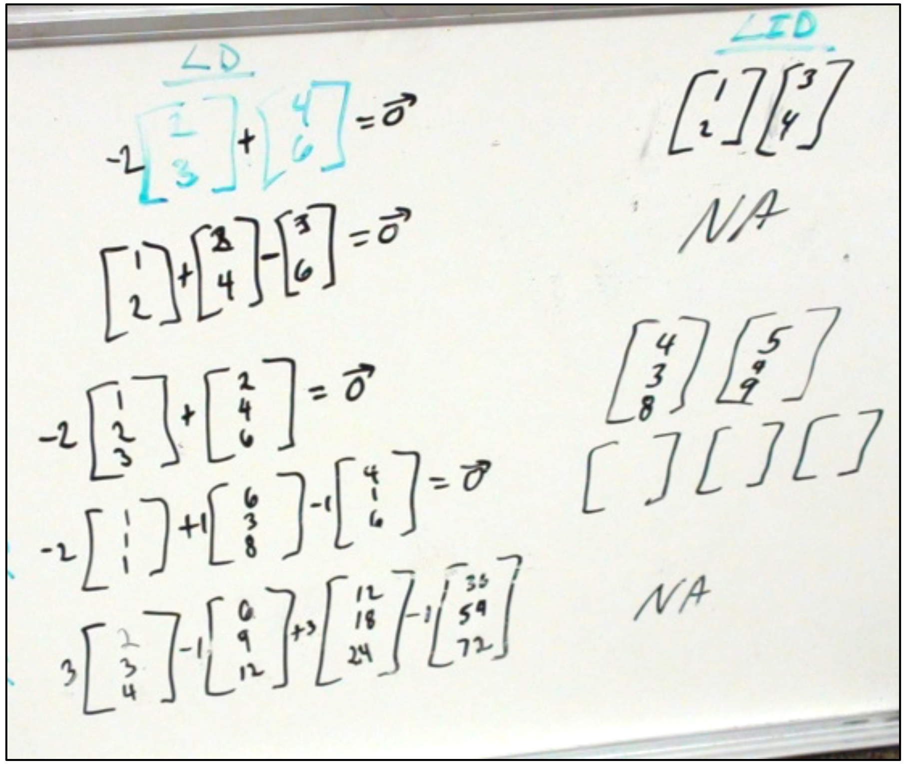
Fill in the following chart with the requested sets of vectors. Keep track of the strategies you use to generate the examples.

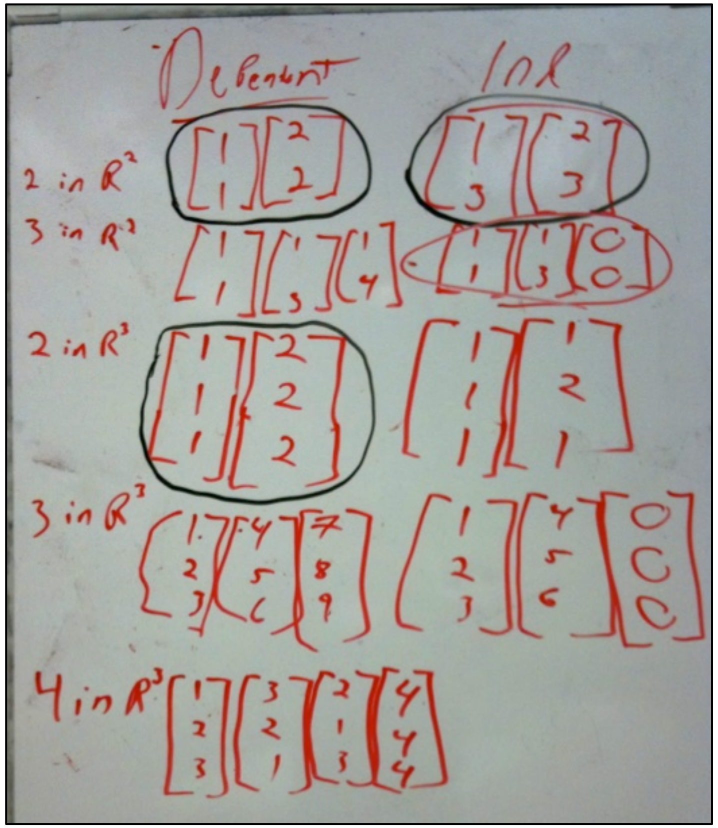
	Linearly dependent set	**Linearly independent set**
A set of 2 vectors in $\mathbb{R}^2$		
A set of 3 vectors in $\mathbb{R}^2$		
A set of 2 vectors in $\mathbb{R}^3$		
A set of 3 vectors in $\mathbb{R}^3$		
A set of 4 vectors in $\mathbb{R}^3$		

Write at least 2 generalizations that can be made from these examples and the strategies you used to create them

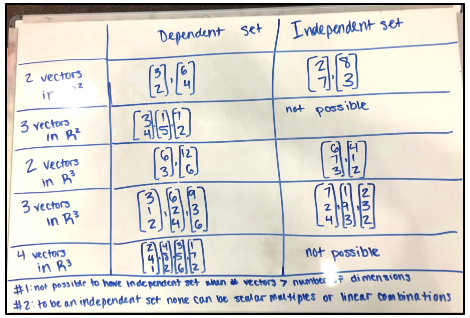
Task 4: Creating Examples & Generalizations

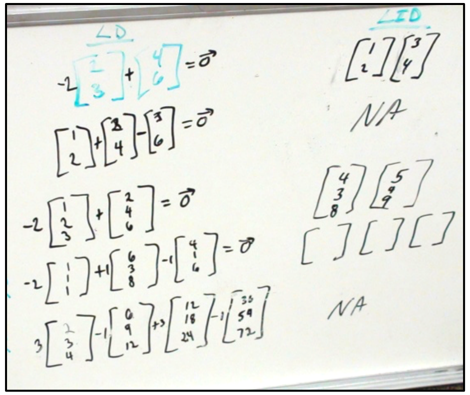


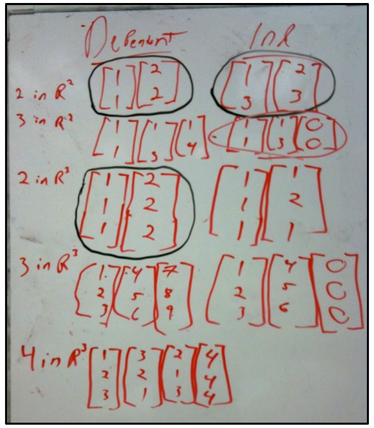




Task 4: Creating Examples & Generalizations







Main Outcomes from Task:

- Students create and justify generalizations about linear (in)dependence beyond $\mathbb{R}^2$ and $\mathbb{R}^3$
  - If a set contains at two vectors that are scalar multiples of each other, then the set is linearly dependent.
  - If at least one vector in a set is a linear combination of the other vectors in the set, then the set is linearly dependent.
  - Any set of vectors in $\mathbb{R}^n$ containing more than n vectors is linearly dependent.
  - Any set containing the zero vector is linearly dependent.
- There is an intellectual need for a way to efficiently check LI or LD; transition from here into row reduction

Summary of Unit 1 in terms of RME

RME Level of Activity	Manifestation in the Magic Carpet Ride Unit
Situational activity involves students working toward mathematical goals in an experientially real setting.	Students explore ways of combining travel vectors in 2D and 3D.
Referential activity involves models-of that refer (implicitly or explicitly) to activity in the original task setting.	Students explore the definitions and new examples of span and linear (in)dependence for sets of vectors through the task setting.
General activity involves models-for that facilitate solutions and interpretations independent of the original task setting.	Students make and support conjectures about properties of sets of vectors regarding linear dependence, linear independence, and span.
Formal activity involves students reasoning in ways that reflect the emergence of a new mathematical reality and thus no longer require support of prior models-for activity.	Students use definitions of span ans linear (in)dependence without needing to unpack the meanings of these definitions (e.g., use those concepts to reason about invertibility or eigentheory).

Current IOLA work

- Create additional IOLA instructional units; create new research findings regarding what is known about student learning in linear algebra; provide professional development for interested instructors
- New units in bold:
  - Unit 1: Linear independence and span
  - ***Unit 2: Solutions to systems of equations***
  - Unit 3: Matrices as linear transformations
  - ***Unit 4: Determinants***
  - ***Unit 5: Subspaces***
  - Unit 6: Change of basis, diagonalization, and eigentheory
  - ***Unit 7: Projection, orthogonalization, and least squares***

(NSF DUE #1915156, 1914793, 1914841)

New Units in Development

Unit 2: System of Linear Equations

Systems Unit

Task 3: Adding constraints and generating examples

Task 3, Part 1: Possible third constraints. The university is considering adding a third constraint, and has asked you to provide feedback on several possibilities. The geogebra server went down during your consulting meeting, and you're asked to share some initial predictions about the corresponding systems and their solution sets. There are a variety of ways in which planes can intersect in three-dimensional space as shown below. WITHOUT using geogebra, provide an initial prediction of which of the following images (a-f) correspond to which systems.

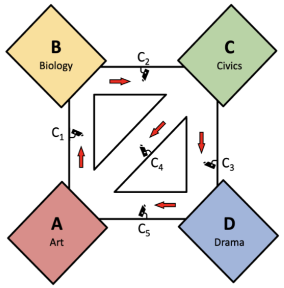
System 1	System 2	System 3	System 4
x+y+z=210 5x+7y+10z=1500 x+y+z=500	x+y+z=210 5x+7y+10z=1500 z=y+20	x+y+z=210 5x+7y+10z=1500 3x+5y+8z=1080	x+y+z=210 5x+7y+10z=1500 3x+5y+8z=800

a. Three planes intersect in one point in R^3. b. Three planes intersect in a line in R^3.

Organized around helping students understand the structure of infinite solution sets to systems of *simultaneous* equations.

Unit 5: Subspaces

TRAVERSING ONE-WAY HALLS IN THE WEST WING



The hallways in one wing of Ida B. Wells High School were changed to one-way corridors to promote social distancing during a pandemic (red arrows in the diagram). These hallways connect classrooms A-D (the Art room, Biology lab, Civics room, and Drama room) as shown in the diagram. Each hall has a security camera that allows Principal McDaniel to monitor student movement through the hallways (cameras 1-5, as shown in the diagram). As a further precaution, each wing is isolated from the rest, so the students in a wing stay within that wing and no students from any other part of the school will enter the west wing.

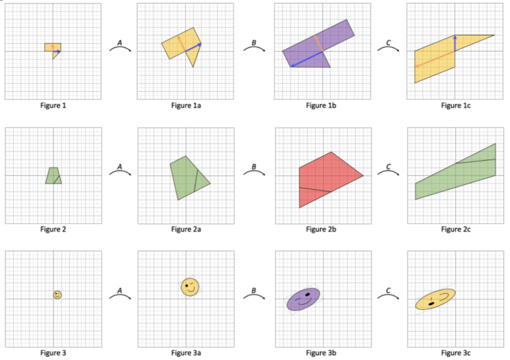
SCENARIO ONE: TRACKING MOVEMENT

Before the school year starts, Principal McDaniel goes into the school when it is empty to spend a day learning how to use data from the camera system. Her daughter Ella comes with her, and she asks Ella to help her test the system by walking between rooms – so long as she stays in the building and follows the one-way hallway rules. Ella decides to start in the Art room, passes Camera 1 (C_1) as she walks from the Art room toward the Biology lab, and then continues walking past Camera 2 (C_2) as she walks from the Biology lab to the Civics room. Afterwards, Principal McDaniel sees that the camera system has recorded the number of people who walked by each camera with the vector $p = \begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}$.

Organized around the notion that subspaces are non-empty subsets of vector spaces that are closed under linear combinations.

Unit 4: Determinants

In these examples, the same matrix transforms all pre-images from one column into their images in the next column. In other words, as you move from one column to the next, it's a single matrix that does the transformation between columns.



Create a systematic way to quantify a distortion of space resulting from a matrix acting on objects in $\mathbb{R}^2$.

- Your goal is to assign a ***single real number*** to any 2x2 matrix to ***measure how it changes the size of objects in space***.
- Your measure needs to work consistently across all 2x2 matrices and all preimage/image pairs. Use the examples above as inspiration.

Organized around understanding determinants as a measure of multiplicative signed change in area/volume caused by the transformation.

Unit 7: Least Squares

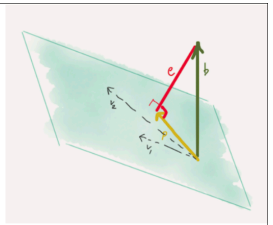
Delivering mail to Gauss in 3D – with a drone

You've got some mail you need to give to Old Man Gauss, who now lives at $\boldsymbol{b} = \begin{bmatrix}-1\\1\\4\end{bmatrix}$.

You still have the same modes of transportation you had before for travel in $\mathbb{R}^3$: $\boldsymbol{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix}6\\3\\8\end{bmatrix}$.

Your cousin has a drone that can travel along *any* vector in $\mathbb{R}^3$. They said they would lend it to you to deliver Gauss's mail -- on the condition that **you get as close as you can** to Gauss's home before you use the drone.

3. **With your group, try to determine the location that you can get to with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ that is the <u>shortest distance</u> to Gauss's house. From there you can use your cousin's drone!**
   a) What is the location that you would travel to in order to use the drone? i.e., what is $\boldsymbol{p}$?
   b) How would you get there with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$? i.e., what are x_1 and x_2?
   c) Along what vector would the drone travel? i.e., what is $\boldsymbol{e}$?
   d) What distance would the drone's trip be? i.e, what is $\|\boldsymbol{e}\|$?



Given info:

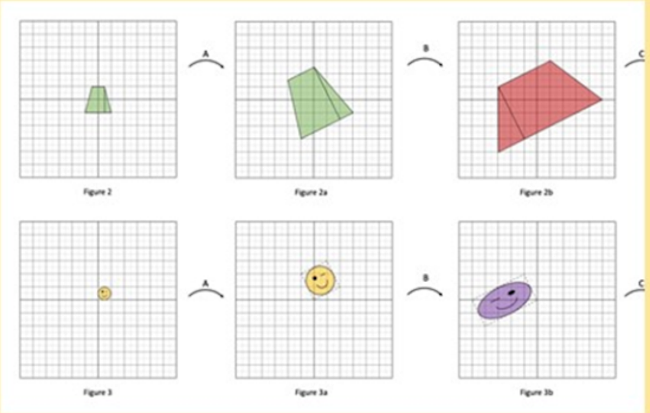
$v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $v_2 = \begin{pmatrix}6\\3\\8\end{pmatrix}$ $b = \begin{pmatrix}-1\\1\\4\end{pmatrix}$

Relationships we determined:

1. $x_1\bar{v}_1 + x_2\bar{v}_2 = \bar{p}$
   if $\hat{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$, $A = \begin{pmatrix}1 & 6\\1 & 3\\1 & 8\end{pmatrix}$
   $A\hat{x} = p$
2. $\bar{p} + \bar{e} = \bar{b}$
   or
   $p = \bar{b} \cdot \bar{e}$
   or
   $\bar{e} = \bar{b} - \bar{p}$
3. $\bar{p} \cdot \bar{e} = 0$
   $\bar{v}_1 \cdot \bar{e} = 0$
   $\bar{v}_2 \cdot \bar{e} = 0$

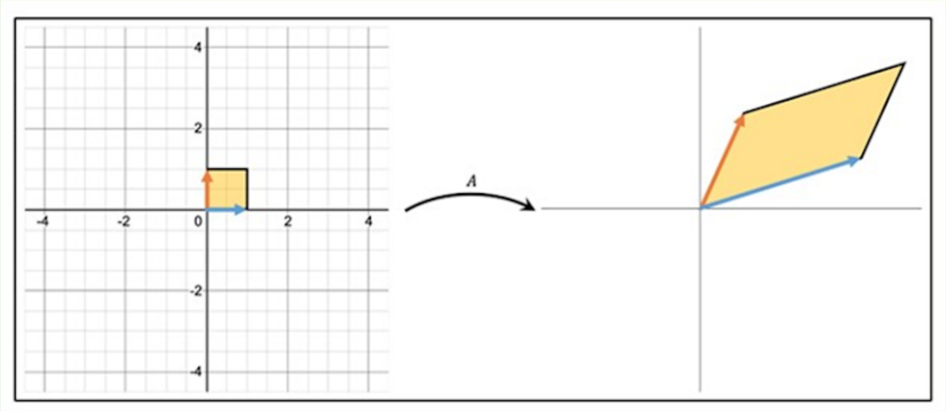
Organized around the closest possible location that one can reach when it is not possible reach a desired location in a vector space.

Determinants Unit



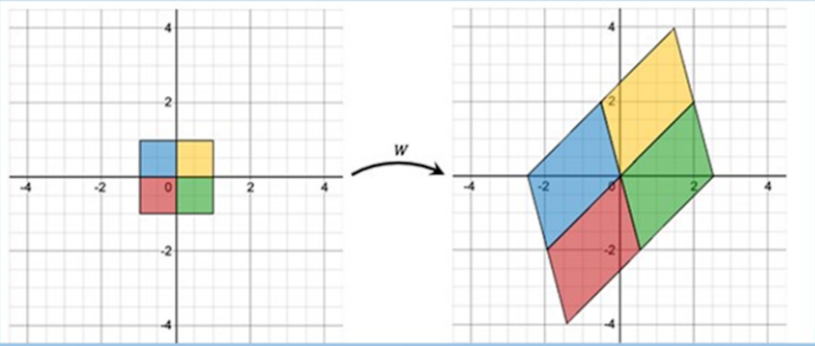
Task 1: Quantifying distortion

Students explore several examples of images distorted by a linear transformation and are asked to identify a single quantity to describe this distortion.



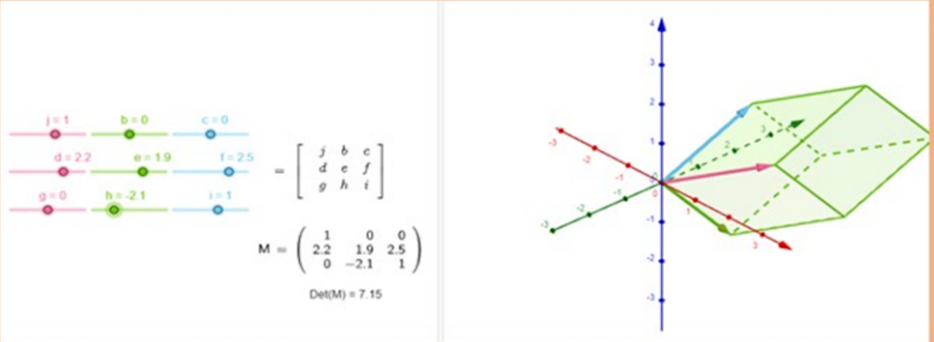
Task 2: 2 x 2 Determinant Formula

Students find a formula for the determinant by inspecting how a unit square is transformed by a generic 2 x 2 matrix.



Task 3: Windows Task

Students are asked to explore how to invert the linear transformation above and identify its effect on the determinant.



Task 4: Exploring Determinants Dynamically

Students explore both 2 x 2 and 3 x 3 matrix transformations dynamically using GeoGebra generating and justifying conjectures involving the determinant.

https://www.geogebra.org/u/iolalinearalgebra

Closing thoughts

- The RME idea of guided reinvention can be an informal, accessible introduction to an idea that later becomes formalized (e.g., ”everywhere you can get” is the idea of span) or a mathematical activity such as conjecturing and proving (e.g., students suggest that sets with the zero vector are LD)
- We have found trying to develop an intellectual need for new concepts or procedural skills to be very helpful for students in following and appreciating the connection of concepts within the course
- Thinking about the teacher as a broker (Lave & Wenger, 1991) between the classroom community and the mathematics community can help imagine your role as fostering students’ mathematical work and then aligning that with formal conventional terms and symbols from the mathematics community (Zandieh, Wawro, & Rasmussen, 2017)

Closing thoughts

- Facilitating a mathematical conversation among the students can be hard and feel unnatural at first
- See "Typical Day" on the IOLA website for suggestions about facilitating whole-class discussion

Productive Whole Class Discussions

To facilitate substantive and rigorous whole class discussions, the following goals are necessary and foundational. Without these, instructors will not have the conditions needed to ensure that the discussion deepens student reasoning and understanding.

- Goal One: Helping Individual Students Share Their Own Thoughts »
- Goal Two: Helping Students Orient to and Listen Carefully to One Another »
- Goal Three: Helping Students Deepen Their Reasoning »
- Goal Four: Helping Students Engage with Others' Reasoning »
- Goal Five: Building on and Extending Students' Ideas »

http://iola.math.vt.edu/typicalday.php

Closing thoughts

- Give students space to explore and pursue a problem-solving approach that is sensible to them
- Encourage/motivate students to engage in each other's ideas and collaborate
- Make listening to and learning from multiple students explaining their approaches a normal part of class
- Ask small groups to collectively share things they are confident about and things they are still confused or curious about in order to encourage openness, curiosity, as well as to honor their current knowledge
- Use alternative modes of interaction. One idea: Discussion Board (credit: Christy Andrews-Larson)
    - Prompt them to summarize their thinking on a task from class (e.g., one generalization about linear independence)
    - Have them share one concept they feel confident they "know"
    - Have them share one thing about which they "wonder"
    - Have them give one "shout out" to someone who influenced their thinking

Thank you!

Look for us at JMM!

Wednesday January 4, 2023, 4:00 p.m.-5:30 p.m.
JMM Workshop: Inquiry-Oriented Linear Algebra: Exploring (infinite) solution sets
Commonwealth, Sheraton Boston Hotel
Organizers:
Christine Andrews-Larson, Florida State University
Michelle Zandieh, Arizona State University
Jessica Lynn Smith, Vanderbilt University
Inyoung Lee, Arizona State University
Minah Kim, Florida State University

Friday January 6, 2023, 4:00 p.m.-5:30 p.m.
JMM Workshop on Inquiry-Oriented Linear Algebra: Exploring Determinants
Commonwealth, Sheraton Boston Hotel
Organizers:
Matthew C Mauntel, Florida State University
Megan Wawro, Virginia Tech
David Plaxco, Clayton State University

McAfee Knob, Appalachian Trail, 45 min from Blacksburg. [Photo Credit]

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