Advanced Integration Application: Probability

Snapshot

Major Concept: Integration is an important technique in computing probabilities and related quantities when a random variable can take on a continuum of possible values.

Before You Begin: Review basic concepts and definitions from the probability section of the textbook.

Standards for Practice and Evaluation: There are many different sorts of questions that you may be asked concerning probability. You will need to be familiar with the terminology, be able to compute relevant quantities, verify that a function is a probability density function, and interpret word problems in probability.

Worksheet Objective

Interpret various terminology from probability theory to correctly set up computations.

Calculate basic quantities like means, medians, and standard deviations as well as more complicated objects like probability density functions.

Predict relationships between means and medians in the presence of tails.

Review the process used to solve each question, making note of connections to earlier sections of the textbook when relevant. Your groups should also discuss what sorts of variations of the given questions might fall within the scope of this topic and/or appear on future exams.

Remember | Understand | Apply | Analyze | Evaluate | Create

Define each of the following concepts in your own words:

- Sample Space
- Random Variable
- Continuous Random Variable
- Discrete Random Variable
- Probability Density Function
- Expected Value

Remember | Understand | Apply | Analyze | Evaluate | Create

- Mean
- Median
- Third Quartile
- First Quartile
- Variance
- Standard Deviation
- Uniform Distribution

Verifying Probability Density Functions

Criteria for PDFs

For f to be a probability density function on an interval (finite or infinite), it needs to be true that
Positive: The function f is nonnegative everywhere on that interval.
Integrable: The function f is integrable (properly or improperly) on every subinterval.
Normalized: The total integral of f must equal 1.

Basic Exam Questions: PDFs

There are two basic question types which rely on the definition of a PDF. The first type will give an example of a function and an interval and ask you to show that the function is (or isn't) a PDF. In this case, it is incumbent on you to **recall and verify each of the three criteria**. For example,

> Show that $f(x) = e^{-x}$ is a PDF on the interval $(0, \infty)$.

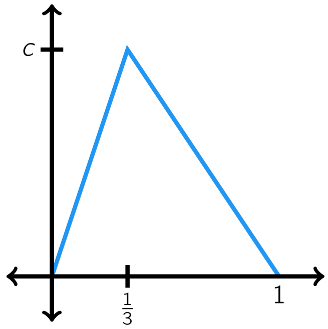
In the second type of question, you may be given partial information about the function or the interval, and you must use these criteria to solve for the missing information. For example,

> The function $f(x) = e^{-x}$ is a PDF on some interval of length 1. What is the interval?

Verifying PDF and Finding Basic Quantities

Remember | Understand | **Apply** | Analyze | Evaluate | Create

The function f graphed below is a piecewise linear function on $[0, 1]$. Find all values of c that make f a probability density function. For this probability density function, what is the full range of values that the random variable might have?



Remember | Understand | **Apply** | Analyze | Evaluate | Create

For the PDF from the question above (using the proper value of c), explain in words what

$$P\left(\frac{1}{3} < X < \frac{1}{2}\right)$$

means and compute its value.

Remember | Understand | **Apply** | Analyze | Evaluate | Create

For the same PDF as above, compute the mean and median of X.

Remember | Understand | Apply | Analyze | Evaluate | Create

A 2009 blog post by the New York Times about LED light bulbs has the following statement: "When it's said that a standard light bulb will last 1,000 hours, that is the mean time to failure: half the bulbs will fail by that point." If the PDF for the lifetime of a standard lightbulb is an exponentially decreasing distribution, has the author used the term "mean" correctly in this sentence? Explain.

`http://bits.blogs.nytimes.com/2009/02/11/how-long-did-you-say-that-bulb-will-last/`

More Advanced Problems and Calculations

Alternate Formula for Variance

If f is the probability density function of a random variable X with values in the interval (a, b) which has mean μ, then

$$\text{Var}(X) = \left(\int_a^b X^2 f(X)\,dX\right) - \mu^2$$

and the standard deviation is the square root of the variance.

Remember | Understand | Apply | Analyze | Evaluate | Create

There are LED light bulbs such that that the probability that a bulb still works after x months of continuous use is equal to

$$\frac{1}{(1+x)^2}.$$

What is the probability density function for the lifetime of such a bulb? What is the expected lifetime of such a bulb and the variance?

Remember | Understand | Apply | Analyze | Evaluate | Create

Give an example of a random variable X which is always finite but has infinite expected value.

Remember | Understand | Apply | Analyze | Evaluate | Create

Give an example of a random variable X whose mean is greater than its median and a second example in which the order is reversed.

The Normal Distribution

Remember | Understand | Apply | Analyze | Evaluate | Create

Give the formula for the PDF of a normally-distributed random variable X with mean μ and standard deviation σ.

Remember | Understand | Apply | Analyze | Evaluate | Create

What are some examples of real-world random variables which are normally distributed or nearly normally distributed?

The Error Function

The **error function** erf is important in applications. For each $z > 0$, $\text{erf}(z)$ is defined as the probability that a normally-distributed random variable with variance $1/2$ takes values in the range $[-z, z]$.

$$\text{erf}(z) = \frac{1}{\sqrt{\pi}} \int_{-z}^{z} e^{-t^2}\, dt.$$

z	$\text{erf}(z)$
0.25	0.2763
0.50	0.5205
0.75	0.7112
1.00	0.8427
1.25	0.9229
1.50	0.9661
1.75	0.9867
2.00	0.9953

Remember | Understand | Apply | Analyze | **Evaluate** | Create

According to weather data from the Franklin Institute, the average high temperature on November 22nd in Philadelphia between 1874 and 2001 was 51.4°F with a standard deviation of 9.5°F. The high temperature on November 22, 2011 in Philadelphia was 61°F. Assuming temperature is normally distributed, what is the probability of it being that warm or warmer?
`https://www.fi.edu/history-resources/philadelphia-weather-data`

Discrete Random Variables

Remember | Understand | Apply | **Analyze** | Evaluate | Create

When two fair dice are rolled, what is the probability that at least one of them lands on 6?

When two fair dice are rolled, what is the median value of the sum of the dice?

Review Guidance

This is the end of our study of techniques and applications of integration.
Be aware that there are many possible questions that you might be asked and ways to use the ideas that we have learned.
Practice exercises from the textbook to become more familiar with some of the other kinds of questions you may be asked on the midterms or the final.