On incorporating research opportunities in undergrad math courses

Pamela E. Harris
Williams College

Outline

- Sketch of history of undergraduate research in mathematics

- Research Experiences for Undergraduates
  - What has changed?
  - What is needed?
  - Where are we headed?

- Moving toward incorporating research as part of undergraduate courses

Directions for Mathematics Research Experiences for Undergraduates

Edited By: Mark A Peterson (*Mount Holyoke College, USA*) and Yanir A Rubinstein (*University of Maryland, USA*)

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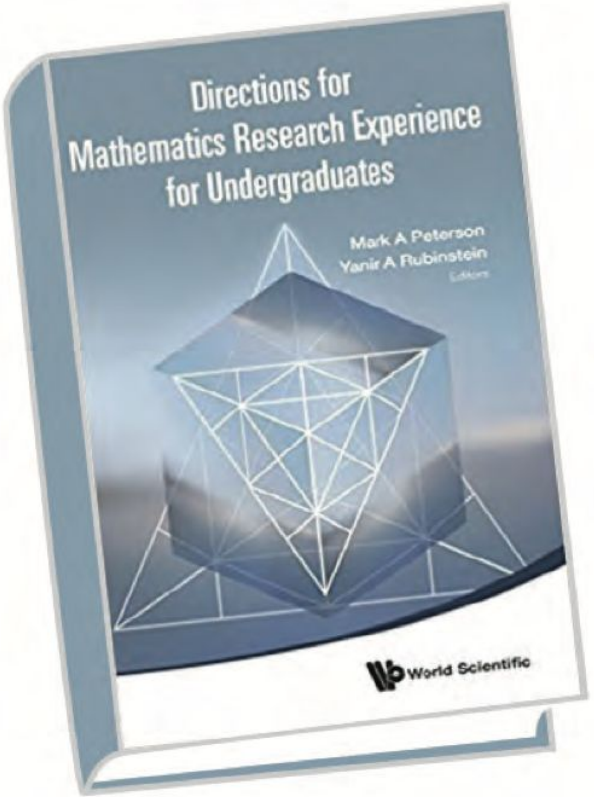
2018 book review by Tamás Forgács

https://www.ams.org/journals/notices/201804/rnoti-p432.pdf



Chapter 1

Undergraduate Research and the Mathematics Profession

Donal O'Shea

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- **1862**: Morrill Act established land grant institutions
- **1869-1899**: Institutions of higher education grow from 563 to 977
  - Universities needed professors and so they establish programs to train them.
- **1900**: US students attend math grad school in Europe

Chapter 1: Undergraduate Research and the Mathematics Profession

- Early 1900's: Math research community experiments with "better" methods of education.
  - 1911: R.L. Moore teaches topology to beginning grad students using a method mimicking research.
- 1934-1935: Michigan Section of MAA appoints committee to create wider interest in undergrad research
- 1942: Frank Griffin surveys college and university mathematics departments to find out which were involving undergraduates in research.

Read at the summer meeting of the MAA, Chicago 1941

UNDERGRADUATE MATHEMATICAL RESEARCH*

F. L. GRIFFIN, Reed College

In the present discussion, however, I am using the word "research" in the more technical sense of exploring some new question or re-exploring some old question and getting results previously unfamiliar to specialists in the field.

UNDERGRADUATE MATHEMATICAL RESEARCH*

F. L. GRIFFIN, Reed College

“[Griffin] documented an impressive array of student research achievements… and noted that colleges that encouraged their undergraduates to do research produced more students who went on to receive graduate degrees in mathematics.”

Donal O'shea

Undergraduate Mathematics

During the first half of the 20th century:

- Some sectional MAA student meetings encouraged student presentation.
- Some mathematics departments required student theses.
- Others had a comprehensive exam: "*a practice almost orthogonal to student research.*"

Post-World War II

- Enrollments in postsecondary institutions
    - More than doubled in the 1960s. Going from 3.6 million enrolled in 1959 to more than 8 million 1969.
    - Tripled from 1960 to 1975, reaching 11.2 million in 1975.
- Institutions increased by 25% in 1960s and another 10% from 1970-1975.

The Sputnik Launch & Election of JFK

“The Sputnik launch by the Soviet Union in 1957 and the subsequent press, not to mention the 1960 presidential campaign, produced a sense that the US needed to catch up by producing more scientists, engineers and mathematicians.” Donal O’Shea

National Science Foundation

- Agency is established in 1950 under the Presidency of Harry S. Truman
- NSF funded some experiments to complement mathematical and science education.
- This included the Undergraduate Research Program
  - Focused mostly on biology, chemistry, and physics
  - Provided students with ten weeks of summer experience in active research labs, research institutions, and some liberal arts colleges

National Science Foundation

In mathematics:

- Projects were more on enrichment activities
- Some accelerated instruction
- Formats often encouraged students to work together and competitively on challenging problems
- One program at Carleton College (directed by Kenneth May) was explicitly aimed at producing undergraduate research

Sentiments on Undergraduate Research in Mathematics

Summary and Resolutions of a 1961 undergraduate research in math conference at Carleton College states:

- “[The aims of undergraduate research] are the training and stimulation of the student, not the attainment of new results, though such bonuses will come occasionally.”

- “Undergraduate research should be judged by standards different from those now employed by mathematicians”

- Acknowledges that “student activity of the research type was a part of good educational practice”

Not much had changed in 20 years!

UNDERGRADUATE MATHEMATICAL RESEARCH*

F. L. GRIFFIN, Reed College

It is, of course, a fair question how generally undergraduates ought to attempt mathematical research. For students who are going on to the graduate school, there is so much fundamental material to be learned as a basis for specialization that it is debatable whether time ought to be used for efforts in research. And whether a student is going on or not, one may wonder how much worthwhile research can be accomplished with the tools available in undergraduate days. Possibly the experience of institutions which have experimented with undergraduate research projects may throw some light on these questions.

Questions and Doubts

- What precisely is “undergraduate research in mathematics”?
- Is it valuable for undergraduate students to do research or is it best to instead master established topics?
- Even with these questions, some faculty continued to encourage students to do research mostly through thesis.
- A missed opportunity on NSF funding!

Joseph Gallian's Program at U of Minnesota Duluth

- 1977: Gallian establishes a summer research program in mathematics at the University of Minnesota Duluth
- 10 week summer program bringing students from across the country to work on unsolved problems.
- Longest running REU!

The 1980s

- The URP program was terminated in 1981.
- The David Report documented a critical shortage of young people entering mathematical careers.
  - National Research Council, Renewing U.S. Mathematics: A Critical Resource for the Future. Washington, DC: The National Academies Press, 1984. (The“David Report.”)
- Congress gave the National Science Foundation a mandate to create a plan to counter the trend.
  - Research Experiences for Undergraduates program began to solicit proposals in 1987.

Current status of REUs

- Currently NSF is funding 38 REU mathematics sites
- Math Institutes host their own REU programs
  - ICERM, MSRI, IPAM, among others
- Professional organizations support undergraduate research
  - MAA NREUP, CURM Mini grants

"Indeed, one could argue that undergraduate research is one of the features characterizing American mathematical education." - Donal O'Shea

REUs are extremely competitive

- SMALL at Williams College
  - On average receives 400-500 applications for 25-45 students
- MSRI UP
  - 2019: Received over 400 applications for 18 students
  - 2021: Received over 250 applications for 18 students
- Summer@ICERM:
  - Receives over 400 applications for ~24 students

Funding Concerns?

“We would do well to remember that federally funded undergraduate research disappeared in 1980 almost overnight.” Donal O’shea

- We have recently seen long established programs received reduced funding or its funding fully eliminated.

Addressing these concerns

- Research experiences in mathematics have become ubiquitous in undergraduate education.
- They have also become extremely competitive and often inaccessible to those who may most benefit from the experience.
- Funding is not secure for the long term.

Proposal

Incorporate research opportunities in math undergraduate courses!

Methods

- Research courses
- Incorporating research in non-research courses
  - Combinatorics - projects
  - Calculus - literature searches and creativity tasks

Undergraduate Research Courses - Tutorials

- Adapted from the Oxford University style of education, the Williams tutorial allows two students to take turns developing independent work.

- The tutorial gives students a sense of ownership of the academic process that inspires ever-greater exploration.

- Research tutorials in representation theory and graph theory.

Tutorial in Representation Theory

- Class was group based and each group had a specific research project that they explored throughout the semester.
- Those with programming experience may contribute to multiple research groups. Two groups of 2, one group of 3.
- Course resulted in 5 publications.

Assessments

Written report

- Collaborative write up of their results.
- Typeset in LATEX.
- Three drafts were submitted.

Final Presentations

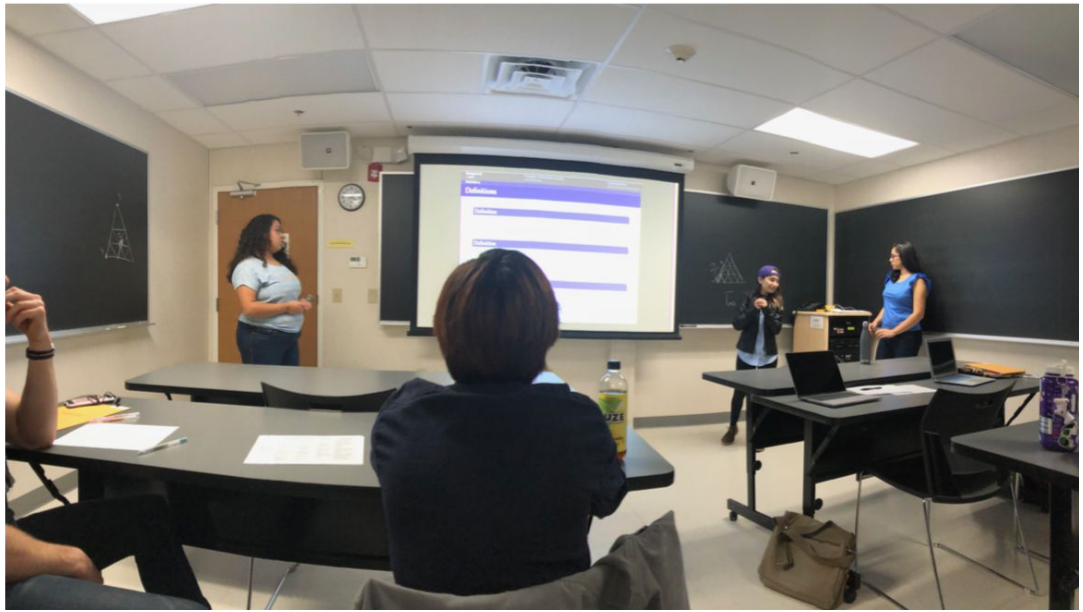
- Each team presented the results of their project.
- 30 minutes long, using slides.

Final presentations were run like a session at a conference.

Tutorial in Graph Theory

We worked through varying projects to find appropriate questions rather that being handed a project directly.

Assessments: We implemented more written drafts and more presentations in front of peers.

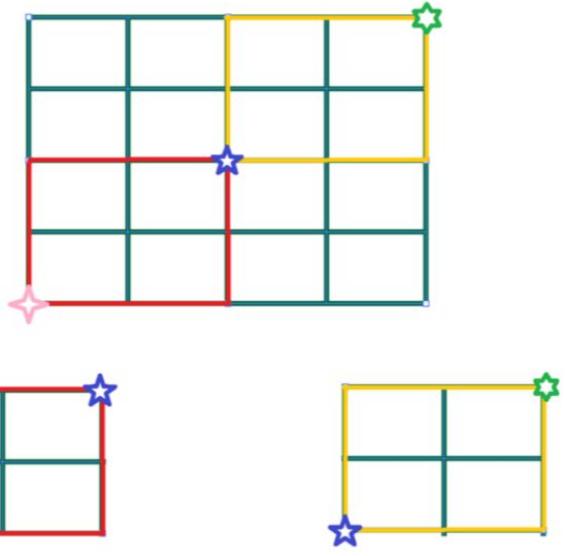


Combinatorics

Introduced some additional topics and students picked to do a deeper undertaking of learning related material.

Topics

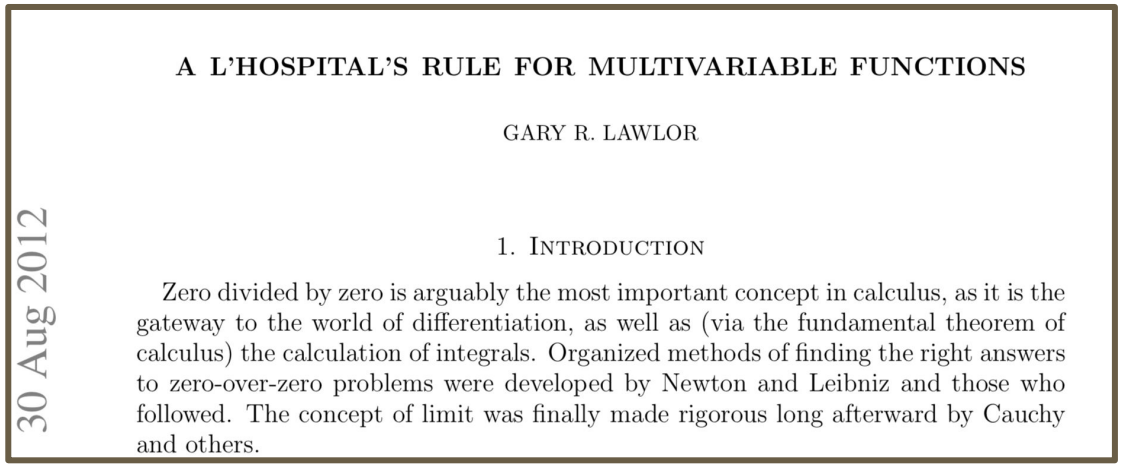
- Trends in parking functions
- Rota’s basis conjecture
- Self-avoiding walks on a lattice
- Vector partition functions



Multivariable Calculus

During pandemic/zoom teaching of calculus I introduced some of my own research in vector partition functions and parking functions.

I showed students the arXiv. To my surprise a student uncovered recent papers on a multivariable version of L'Hopital's rule!



A L'HOSPITAL'S RULE FOR MULTIVARIABLE FUNCTIONS

GARY R. LAWLOR

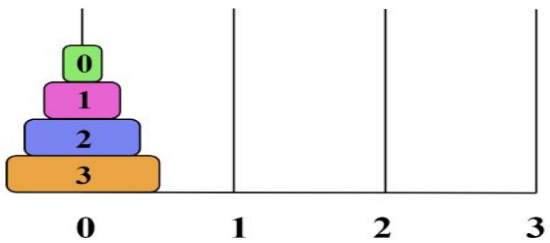
30 Aug 2012

1. Introduction

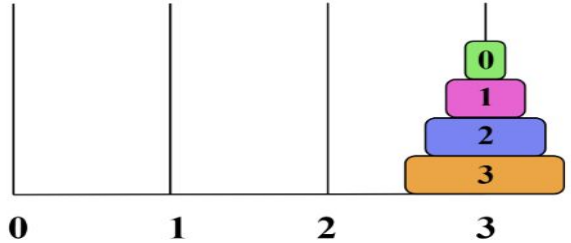
Zero divided by zero is arguably the most important concept in calculus, as it is the gateway to the world of differentiation, as well as (via the fundamental theorem of calculus) the calculation of integrals. Organized methods of finding the right answers to zero-over-zero problems were developed by Newton and Leibniz and those who followed. The concept of limit was finally made rigorous long afterward by Cauchy and others.

Multivariable Calculus

- Three of the students in that class reached out to do research with me.
- We worked on parking functions and we have submitted one paper and motivated research questions for the 2021 MSRI UP summer students.



(a) Starting position.



(b) Ending position.

- One of these students continues to work with me while another is serving as a TA for my current calculus course.

Place for Growth - Collaborative Teamwork

I implemented community agreements with written expectations and ways to reach out when things went wrong. Also self/peer-evaluations:

Turning Student Groups into Effective Teams

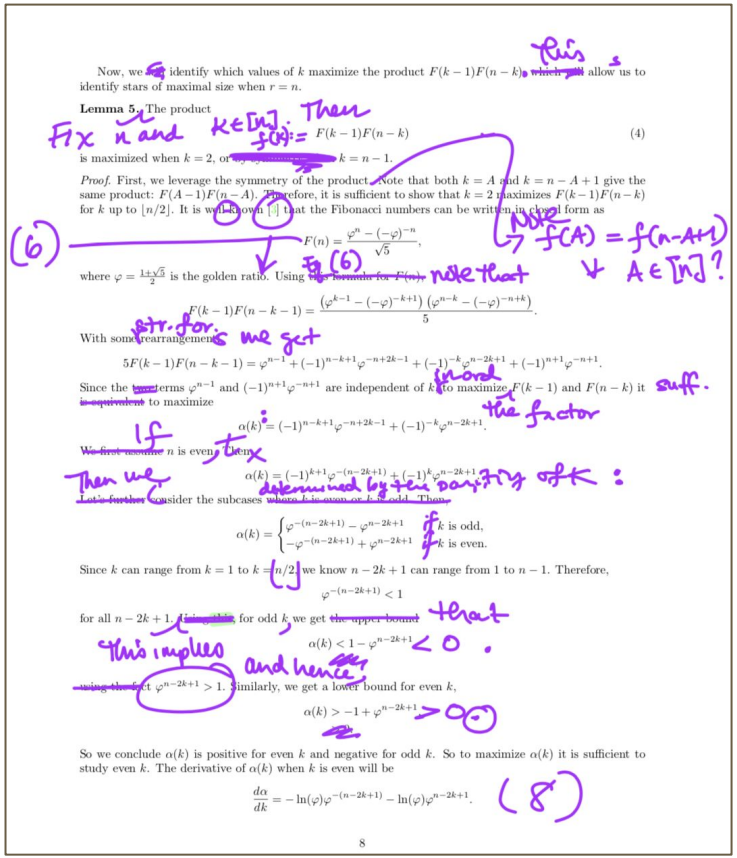
Oakley, et. al. Journal of Student Centered Learning,Volume 2, No. 1, 2004.

Next time I plan to implement assigned readings on teamwork, including:

- Robert Sutton. 2006. “Eight tips for Better Brainstorming.”
- Andy Ko "How to be Creative."
- Katzenbach and Smith (1993). The Discipline of Teams.
- Pausch. "Tips for Working Successfully in a Group."

Feedback on Written Work

- I provided extensive feedback on students' written work.
- Extremely time consuming, luckily courses were small.
- During pandemic/zoom time I began to implement video feedback!



Now, we identify which values of k maximize the product $F(k-1)F(n-k)$, which will allow us to identify stars of maximal size when $r = n$.

Lemma 5. The product

$$F(k-1)F(n-k) \tag{4}$$

is maximized when $k = 2$, or $k = n-1$.

Proof. First, we leverage the symmetry of the product. Note that both $k = A$ and $k = n - A + 1$ give the same product: $F(A-1)F(n-A)$. Therefore, it is sufficient to show that $k = 2$ maximizes $F(k-1)F(n-k)$ for k up to $\lfloor n/2 \rfloor$. It is well known [4] that the Fibonacci numbers can be written in closed form as

$$F(n) = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Using this formula for $F(n)$, note that

$$F(k-1)F(n-k-1) = \frac{\left(\varphi^{k-1} - (-\varphi)^{-k+1}\right)\left(\varphi^{n-k} - (-\varphi)^{-n+k}\right)}{5}.$$

With some rearrangement we get

$$5F(k-1)F(n-k-1) = \varphi^{n-1} + (-1)^{n-k+1}\varphi^{-n+2k-1} + (-1)^{-k}\varphi^{n-2k+1} + (-1)^{n+1}\varphi^{-n+1}.$$

Since the terms φ^{n-1} and $(-1)^{n+1}\varphi^{-n+1}$ are independent of k, to maximize $F(k-1)$ and $F(n-k)$ it is equivalent to maximize

$$\alpha(k) = (-1)^{n-k+1}\varphi^{-n+2k-1} + (-1)^{-k}\varphi^{n-2k+1}.$$

We first assume n is even, then

$$\alpha(k) = (-1)^{k+1}\varphi^{-(n-2k+1)} + (-1)^{k}\varphi^{n-2k+1}$$

Let's further consider the subcases where k is even or k is odd. Then

$$\alpha(k) = \begin{cases} \varphi^{-(n-2k+1)} - \varphi^{n-2k+1} & \text{if } k \text{ is odd,} \\ -\varphi^{-(n-2k+1)} + \varphi^{n-2k+1} & \text{if } k \text{ is even.} \end{cases}$$

Since k can range from $k = 1$ to $k = \lfloor n/2 \rfloor$ we know $n - 2k + 1$ can range from 1 to $n - 1$. Therefore,

$$\varphi^{-(n-2k+1)} < 1$$

for all $n - 2k + 1$. For odd k, we get

$$\alpha(k) < 1 - \varphi^{n-2k+1} < 0.$$

using the fact $\varphi^{n-2k+1} > 1$. Similarly, we get a lower bound for even k,

$$\alpha(k) > -1 + \varphi^{n-2k+1} > 0.$$

So we conclude $\alpha(k)$ is positive for even k and negative for odd k. So to maximize $\alpha(k)$ it is sufficient to study even k. The derivative of $\alpha(k)$ when k is even will be

$$\frac{d\alpha}{dk} = -\ln(\varphi)\varphi^{-(n-2k+1)} - \ln(\varphi)\varphi^{n-2k+1}.$$

8

Figure 4: An undirected graph and its Laplacian matrix[3]

number is bounded by $|M(G)| - |S(G)|$. That is, the highest accessibility number that any transient can have is the size of the set of all transients. Hence, in the case of c transient, we know $A(c) \leq (|M(G)| - |S(G)|)$. This leads to the following.

Definition 6. If $A(c) = |M(G)| - |S(G)|$, then c is transient and it is accessible by every other transient. A transient c with this property is called a **super accessible transient (SAT)**.

In this paper, we ~~seek~~ to characterize the existence of SATs for all sandpile graphs, and to find a formula for them ~~in the cases that they do~~ exist. To do so, we need some additional background.

2.2 Configurations, Burning and Firing

We begin by defining some general graph theory terminology in order to further our study of sandpile monoids.

Definition 7. The **Laplacian matrix** of a graph G, denoted L, is the $|V| \times |V|$ matrix defined as

$$L = \text{diag}(\text{outdegree}(G) - \text{Adj}(G))$$

where the diag(outdegree(G)) is the diagonal matrix consisting of the outdegrees of the vertices of V and Adj(G) is the adjacency matrix of G, where entry (i, j) is equal to the number of edges from i to j. We should underscore that the order of i and j matters here in that (i, j) represents directed edges from i to j and not the other way around.

Now we turn towards vocabulary specific to the study of sandpiles.

Definition 8. The transposed reduced Laplacian matrix $\tilde{L}$ of a sandpile graph G is the $\times$ matrix

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Note that in Figure 5, each of the vertices in the initial configuration is stable, so there does not exist a legal firing for any of them, as the vertices don't need to topple on the sandpile c. Nevertheless, if we consider the firing vector $(1 \quad 1 \quad 1)^T$ as a whole, we see that it is indeed a legal firing vector since the resulting configuration has nonnegative entries.

Definition 10 (Adapted from p. 7, [1]). Let S be the sequence of vertices $(v_{z_1}, v_{z_2}, \ldots v_{z_k})$ where each

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To start, start small, but start you must!

- Talk to students about your research.
- Show them that mathematics is an active field of study.
- Assign research papers that go beyond the usual course material.
- Build small projects into your courses or open ended questions which students can explore collaboratively.
- When you are ready, or even when you are not, build research courses!

Focus on Development Areas

1. Research comprehension and communication skills
2. Practical research skills
3. Research ethics
4. Researcher identity
5. Researcher confidence and independence
6. Equity and inclusion awareness and skills
7. Professional and career development skills

*Center for the Improvement of Mentored Experiences in Research (CIMER) and the Wisconsin Institute for Science Education and Community Engagement (WISCIENCE)

Creating/Finding Open Problems

A number is happy if the sum of the squares of its digits iterates to 1.

- 19 → 82 → 68 → 100 → 1
  So 19 is a 2-power 10-happy number
- 4 is not!

- Work expands to other bases and other exponents.
- Recently, we considered the factoradic representation and small exponents.
- **What else could you consider?**

Students Developing Accessible Research Projects

- Allow students to choose a problem they find especially interesting in your course.
- Ask them to brainstorm a question without too much guidance.
- The student completes a literature search.
    - Find the papers, skimming them, choosing to read more in-depth, and then summarizing them.
- Repeat this process a few times and then the student can choose their favorite to focus on.

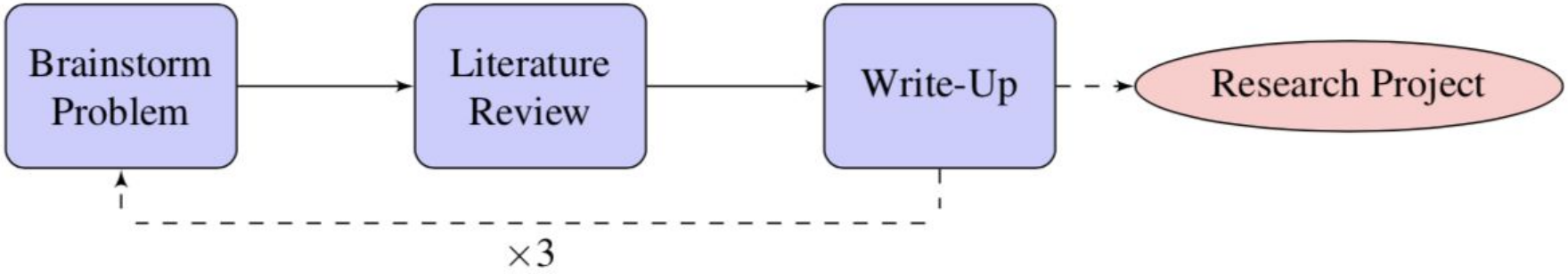
Florida Gulf Coast University Honors Contract

Inspiration to what such a proposal might entail.

“The student will practice learning to think like a mathematician by

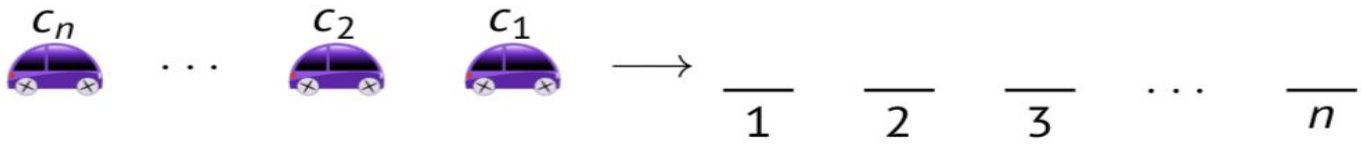
1. developing her own questions that build on material learned in class,
2. researching what is currently known about the question through a literature
   search, and
3. writing up a summary of findings to share with the class.”

Florida Gulf Coast University Honors Contract



An alternative method to the standard undergraduate research project.

Parking Functions



- A one-way street has n parking spots numbered 1 through n.
- n cars enter the street, one at a time, each having a preference for a parking spot.
- Parking Rule:
  - Each car tries to park in its preferred spot.
  - If occupied, then the car parks in the next available spot moving forward in the street.

Parking Functions

As part of her undergraduate studies at Florida Gulf Coast University, Alyson developed a new type of parking function problem which allowed cars to back up and park in a spot behind their preferred parking spot if they found that occupied.

Alyson Baumgardner

Dr. Katie Johnson

This question has now motivated three summers of undergraduate math research projects involving ~40 students!

Reach out to your friends!

Ayo Adeniran

Jessica De Silva

Alexander Diaz-Lopez

Christina Eubanks-Turner

Luis David Garcia Puente

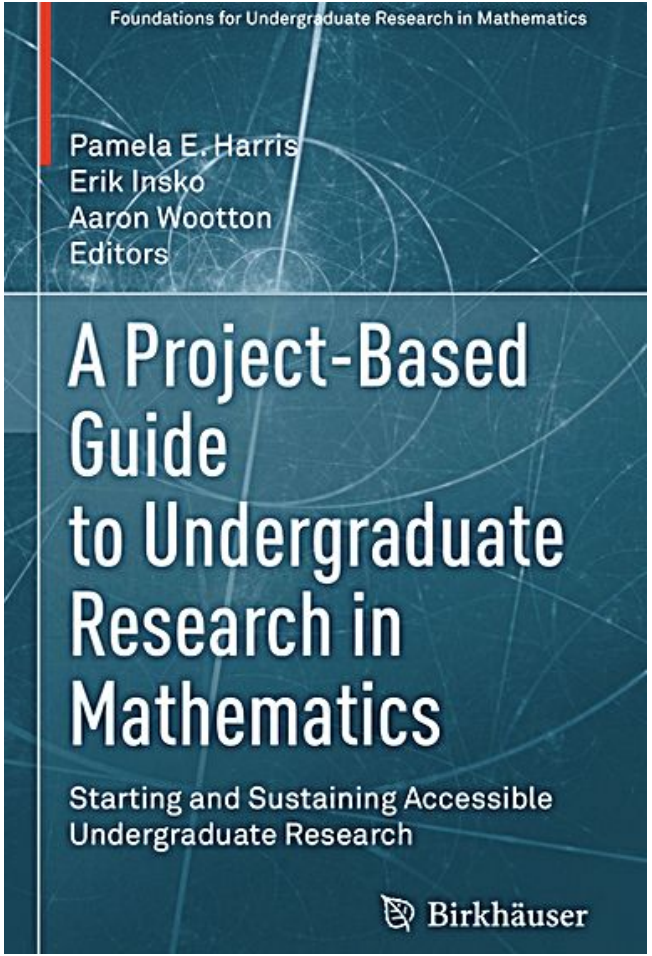
They provided open problems for the 2021 Graph Theory Research Course!

Book with many ideas!

Provides accessible and self-contained research problems designed for undergraduate student projects, and promotes the development of sustainable undergraduate research programs.

Research areas include:

- Mathematics education
- Disease modeling
- Tropical curves and surfaces
- Numerical semigroups
- Graph theory



Chapter 7: Lateral movement in undergraduate research

Stephan Ramon Garcia

Abstract We explore the thought processes, strategies, and pitfalls involved in entering new territory, developing novel projects, and seeing them through to publication. We propose twenty-one general principles for developing a sustainable undergraduate research pipeline and we illustrate those ideas in three case studies.

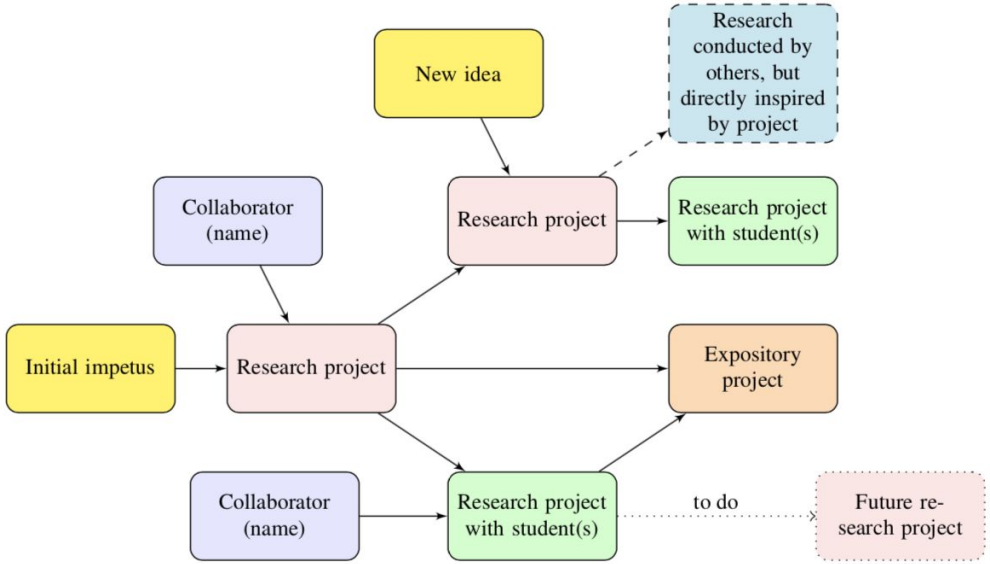


Fig. 1: Sample project flowchart. Time flows approximately from left to right. Initial problems or external inflows of knowledge are in yellow. Collaborators coming on board are in purple.

Research Experiences for Undergraduate Faculty

Research Experiences for Undergraduate Faculty (REUF) is a program for undergraduate faculty who are interested in mentoring undergraduate research.

American Institute of Mathematics

ICERM

Thank you!

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