A Radical Approach To Calculus

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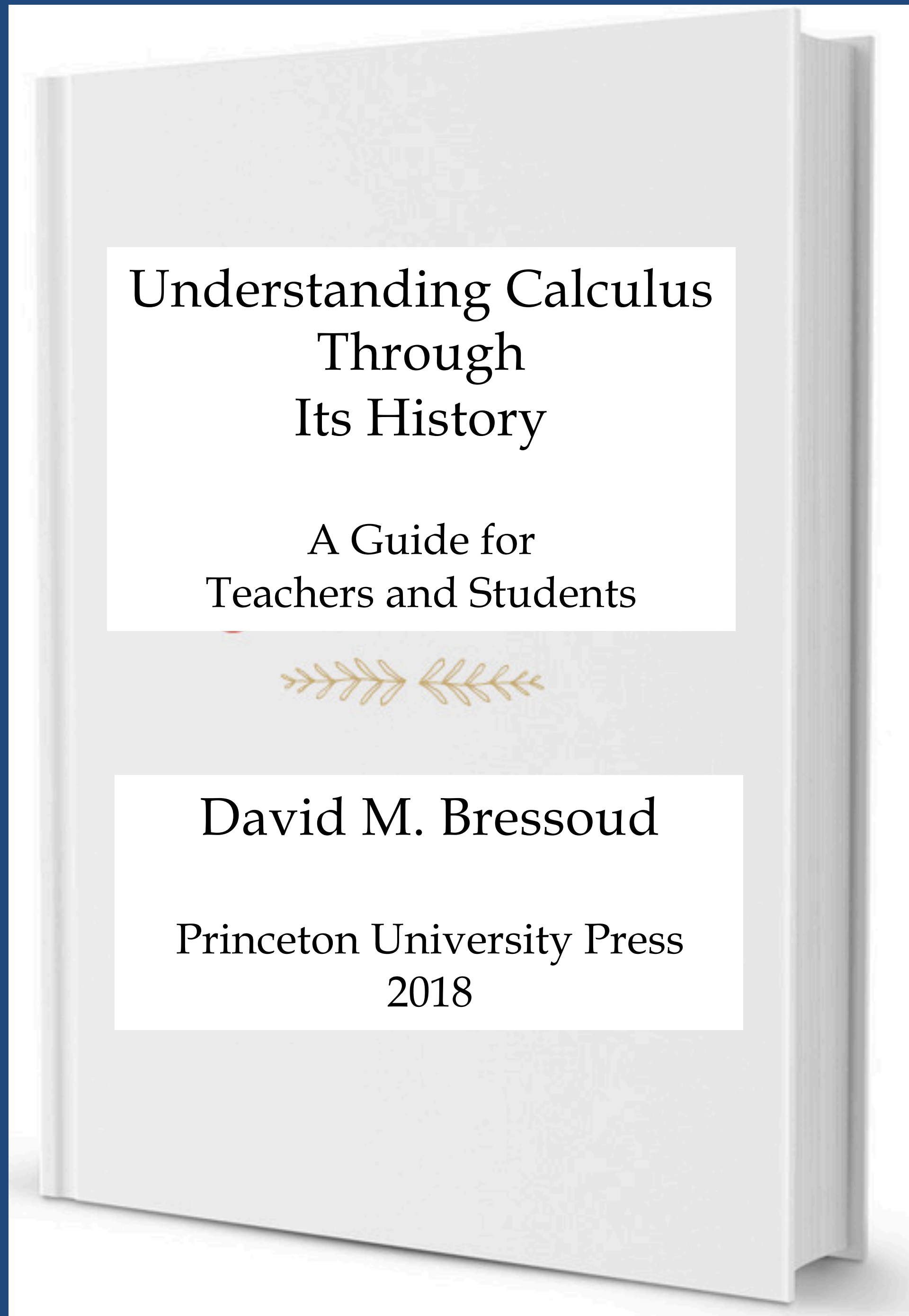
A pdf file of this PowerPoint is available at www.macalester.edu/~bressoud/talks


Understanding Calculus
Through
Its History
A Guide for
Teachers and Students
David M. Bressoud
Princeton University Press
2018

NCTM Research Compendium, to appear this year

19 Understanding the Concepts of Calculus: Frameworks and Roadmaps Emerging From Educational Research

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Calculus is a foundational course for most disciplines in science and engineering around the world. It lies at the heart of any modeling of dynamical systems and often is used to signal whether a student is prepared for advanced mathematics, science, and engineering, even when such courses do not explicitly build on calculus (Bressoud, 1992). At the same time, calculus is a barrier to the academic progress of many students. Across the United States, 28% of those enrolled in postsecondary calculus 1 (typically consisting of differential calculus) receive a D or F or withdraw from the course (Bressoud, Carlson, Mesa, & Rasmussen, 2013). Only half earn the B or higher that is taken as a signal that one is prepared for the next course, and many of these, despite their grade, are discouraged from continuing (Bressoud et al., 2013). New challenges have arisen, from the movement of calculus ever earlier into the secondary curriculum in the United States to the pressure to drastically reduce failure rates (Bressoud, 2015). Meeting these challenges will require the research community to develop better understandings of how students negotiate this subject, where the pedagogical obstacles lie, and what can be done to improve student success.

In the interest of assuring the coherence of this chapter, and to provide an appropriate level of detailed attention to the work we discuss, we concentrate our attention on the research focused on students' understanding of calculus content. However, we are compelled to first acknowledge the wide variety of important educational research that has been done on other issues related to calculus. For example, the recent national study by the Mathematical Association of America (Bressoud, Mesa, & Rasmussen, 2015) focused on identifying characteristics of college calculus programs that contribute to student success as measured by retention and changes in attitudes. Other work has explored issues related to the rapid growth of the Advanced Placement Calculus program in the United States (Keng & Dodd, 2008; Morgan & Klaric, 2007). Törner, Potari, and Zachariades (2014) provide an overview of curricular evolution in calculus in Europe at the secondary level. There has also been research on students' readiness to learn calculus (Carlson, Madison, & West, 2015). Finally, there has been research focused on calculus instructors. This work includes investigations focused on instructors' perceptions of instructional approaches (Sofronas et al., 2015), relationships between teaching practices and content coverage concerns (Johnson, Ellis, & Rasmussen, 2015), and the professional development of graduate students (Deshler, Hauk, & Speer, 2015).

Schoenfeld (2000) noted that research in mathematics education has two purposes. The first is a pure research purpose, "To understand the nature of mathematical thinking, teaching, and learning," and the second is an applied purpose, "To use such understandings to improve mathematics instruction" (p. 641). It makes sense to organize this chapter around these two purposes for two reasons. First, such an organization will allow us to explicitly shine a light on applied research. It is critical that we do so because calculus is a key part of science, technology, engineering, and mathematics

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Traditional order of four big ideas:

1. Limits:
2. Derivatives:
3. Integrals:
4. Series:

Problems

1. Limits: as x approaches c, $f(x)$ approaches L

- Leads to assumption that f cannot oscillate around or equal L when $x \neq c$
- x-first emphasis makes transition to rigorous definition difficult
- Difficult to prove theorems that rely on definition of limit
- Belief that if $\lim_{x\to a} f(x) = b$ and $\lim_{y\to b} g(y) = c$, then $\lim_{x\to a} g\big(f(x)\big) = c$

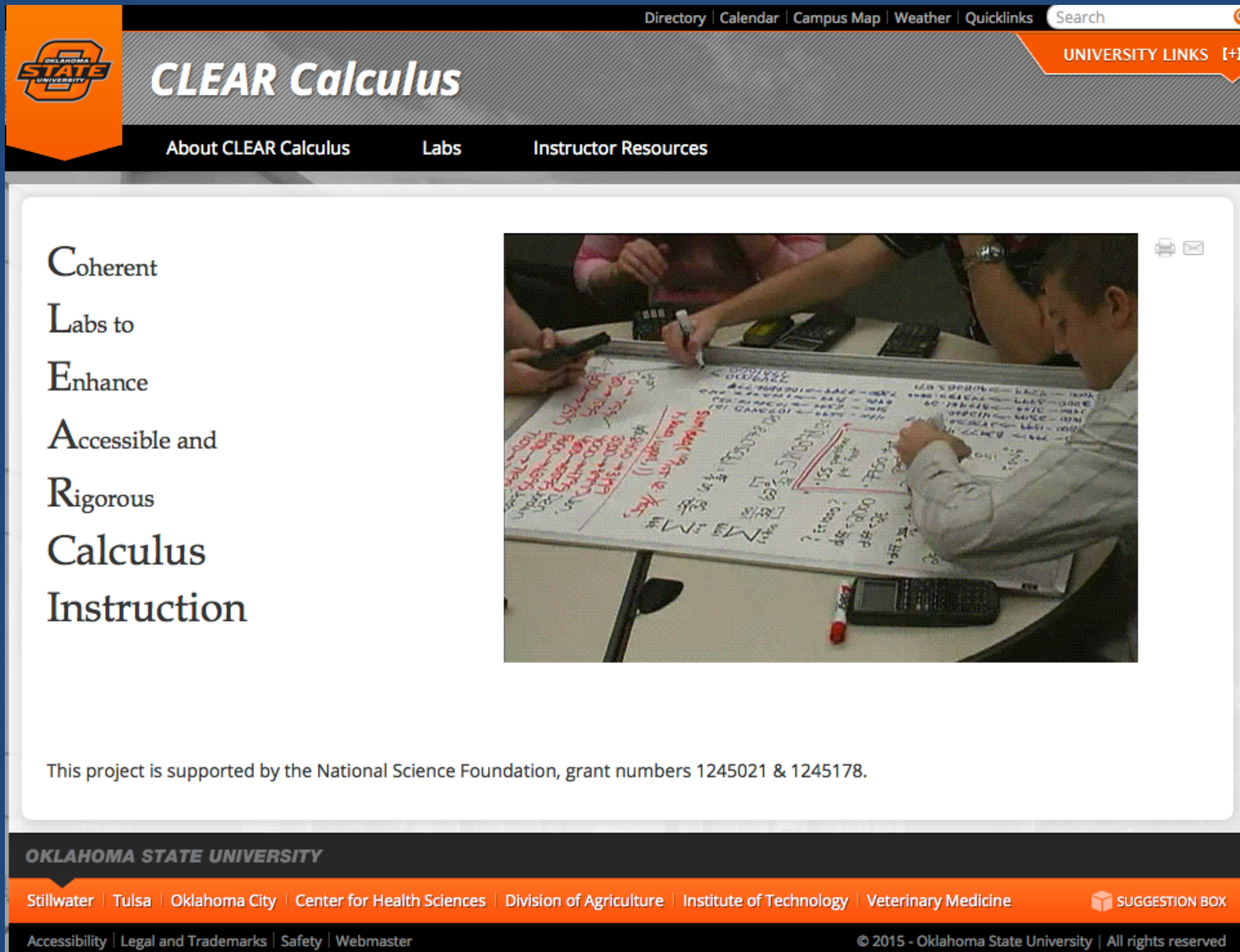
Solution

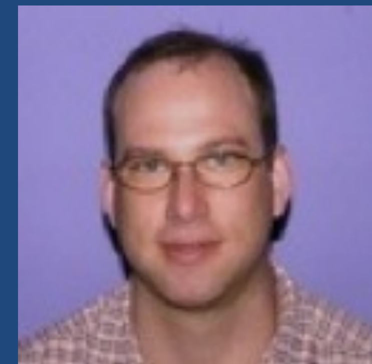
1. Limits: Algebra of Inequalities

Build from bounds on approximations

Leibniz series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$

Justified because each partial sum differs from $\frac{\pi}{4}$ by less than absolute value of next term.





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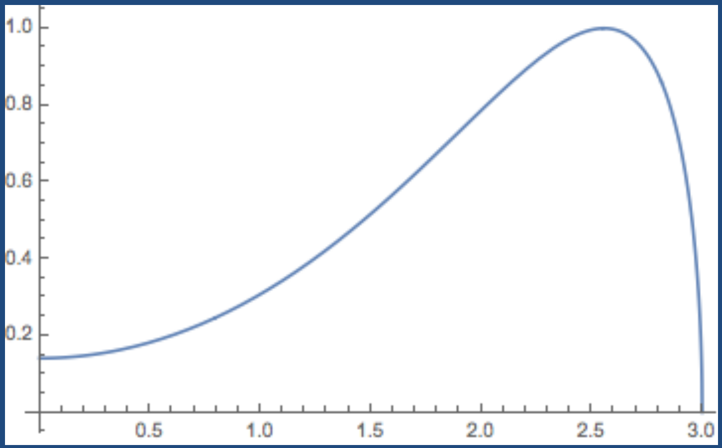
Lab 12: Example (distance-time-velocity)

In Lab 7, we were given information about the NASA Q36 Robotic Lunar Rover. Specifically, it can travel up to 3 hours on a single charge and has a range of 1.6 miles. After t hours of traveling, its speed is $v(t)$ miles per hour given by the function $v(t) = \sin\sqrt{9-t^2}$. One hour into a trip, the Q36 will have traveled 0.19655 miles. Two hours into a trip, the Q36 will have traveled 0.72421 miles. Now we'll see how to calculate values for the distance traveled as a function of time, even though we won't be able to write that function in terms of elementary functions.

In Lab 7, we computed the acceleration $a(t) = v'(t) = \dfrac{-t\cos\sqrt{9-t^2}}{\sqrt{9-t^2}}$ which is positive from $t = 0$ to $t = \sqrt{9-\pi^2/4} \approx 2.556$.

Let's approximate the distance traveled by the Q36 in the first two hours.

$$v(t) = \sin\sqrt{9-t^2}$$



We could get quick, but not very accurate, approximations by pretending that the Q36 traveled at a constant speed the entire time. Since the rover is always speeding up during these two hours, using the initial and final speeds would produce an underestimate and overestimate for the distance traveled, respectively:

Underestimate: 2 hours at $v(0) = 0.14112$ mph is 0.28224 miles. Overestimate: 2 hours at $v(2) = 0.78675$ mph is 1.57350 miles.

This is a huge range, so using either of these as an approximation, the best we can say is that we are within $1.57350 - 0.28224 = 1.29126$ miles of the exact answer. Of course we were told that the rover travels 0.72421 miles in the first two hours, so in this case we can compute the exact errors (usually not possible, otherwise you wouldn't be approximating):

The underestimate 0.28224 miles is $|0.28224 - 0.72421| = 0.44197$ miles off.

The overestimate 1.57350 miles is $|1.57350 - 0.72421| = 0.84929$ miles off.

Although not very accurate, both of these errors are smaller than our computed error bound:

$$\text{error} < 1.29126 \text{ miles.}$$

Neither of these approximations is exact because the Q36 is not traveling at a constant speed, thus simply using $d = vt$ isn't sufficient.

Problems

2. Derivatives: slope of tangent

- Derivative becomes a static number
- Students have difficulty making the connection to average rate of change
- Makes it difficult to understand derivative as relating rates of change of two connected variables

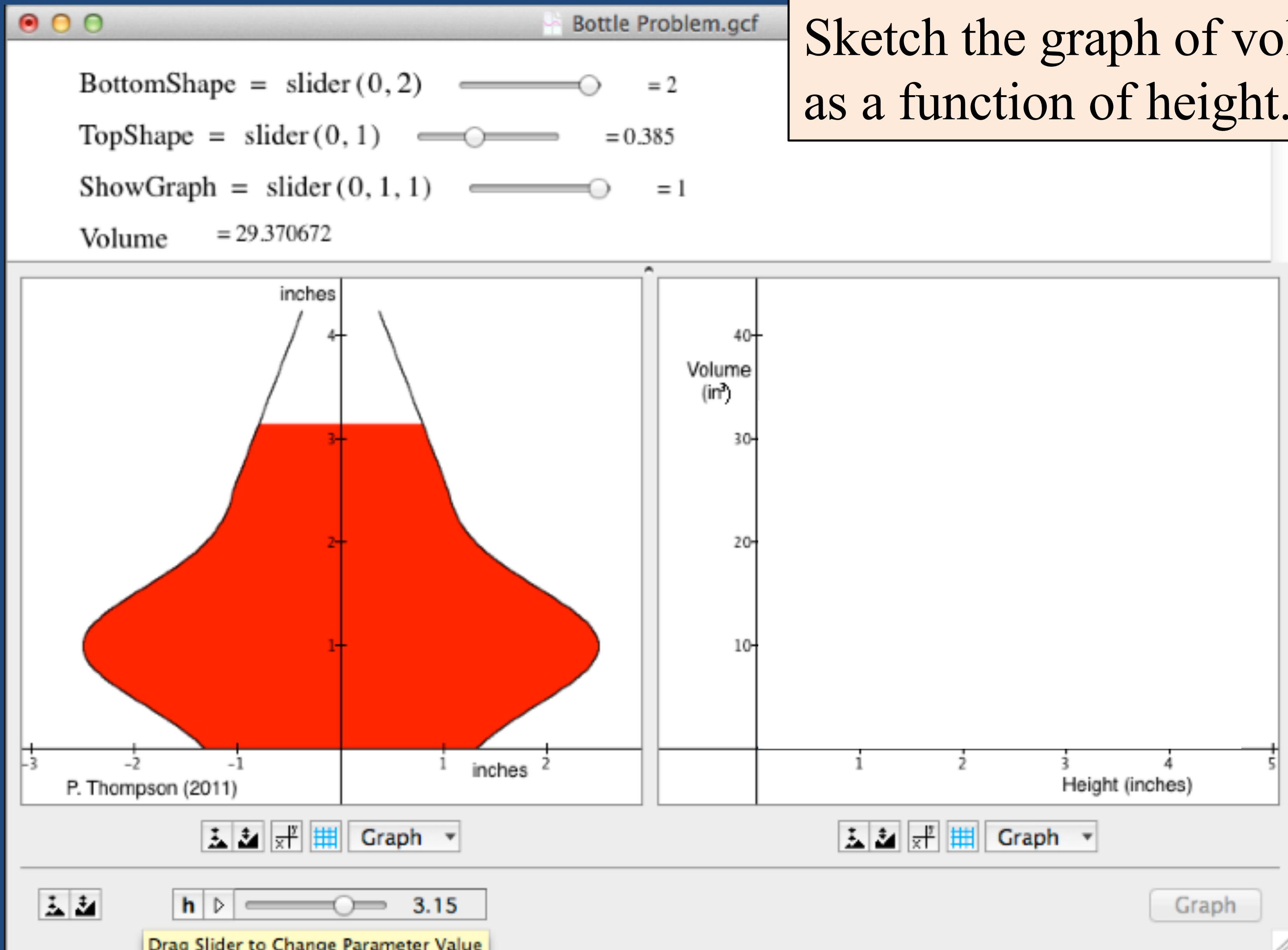
Solution

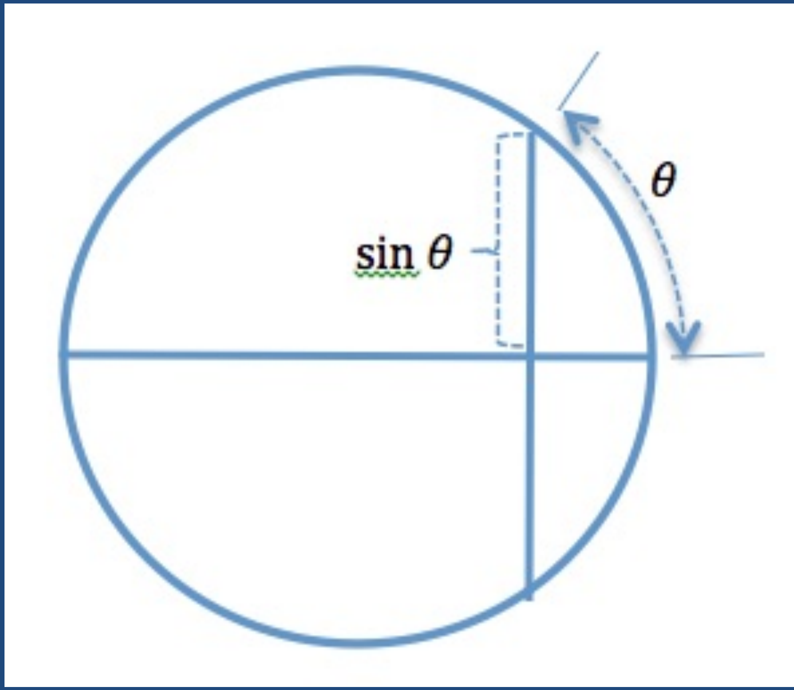
2. Derivatives: Ratios of Change

Focus on function as a relationship between two linked variables

Derivative connects small changes in one to small changes in the other

Sketch the graph of volume as a function of height.





Indian astronomy:
Arclength θ measured in minutes

Circumference

$= 60 \cdot 360 = 21{,}600$

Radius = 3438

~ AD 500, Aryabhatta showed that for small increments

$$\frac{\Delta\ \text{sine}}{\Delta\ \text{arclength}} \sim \cos\theta$$

Problems

3. Integrals: area under curve

- Students don’t see integral as accumulator
  “I don’t understand how a distance can be an area.”
- Leads to difficulties interpreting definite integral with variable upper limit, critical to understanding the Fundamental Theorem of Integral Calculus
- Don’t retain definition of definite integral as limit of Riemann sums

Wagner, J.F. (2017). Students' obstacles to using Riemann sum interpretations of the definite integral

1st-year physics students see Riemann sums as either irrelevant or simply a tool for approximating definite integrals.

3rd-year physics majors cannot justify why the following produces the area under $y = x^3$ from 0 to 2.

$$\int_0^2 x^3\,dx = \frac{1}{4}x^4\bigg|_0^2 = \frac{16}{4} - 0 = 4.$$

See launchings.blogspot.com April, 2018

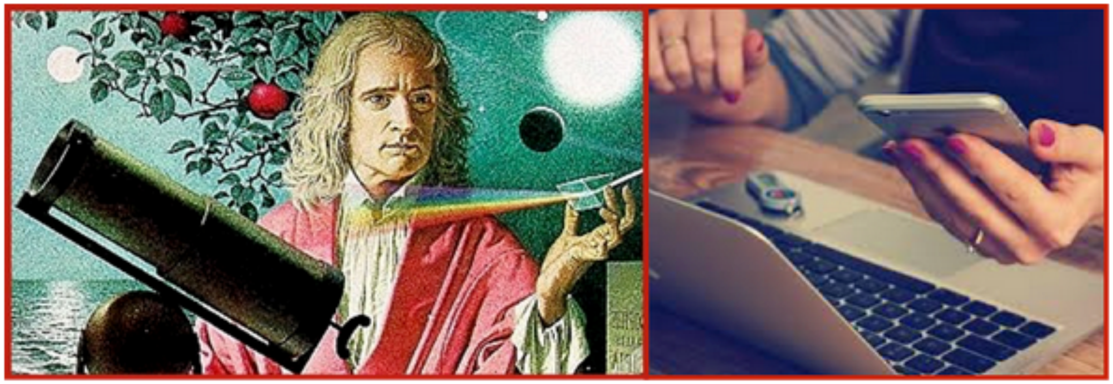
Solution

3. Integrals: Accumulation

START with accumulator functions, *i.e.* Riemann sums with variable upper limit, leading to $\int_0^x t^3 \, dt$. This accumulates up to x the quantity whose rate of change is t^3.

Students are easily led to discover that rate of change of this function is x^3. Leads to FTIC.

Calculus: Newton Meets Technology



A textbook emanating from

Project DIRACC: Developing and Investigating a Rigorous Approach to Conceptual Calculus

Patrick W. Thompson, Mark Ashbrook

http://patthompson.net/ThompsonCalc/

Problems

4. Series: Infinite Summations

- Students view series as sums with a LOT of terms
- Convergence tests become arcane rules with little or no meaning

Solution

4. Series: Sequences of Partial Sums

Taylor polynomials rather than Taylor series

Prefer emphasis on Lagrange error bound (as extension of Mean Value theorem) rather than convergence tests.

$$f(x) = f(a) + f'(a)(x - a) + E(x, a)$$

$$E(x, a) = \frac{f(x) - f(a)}{x - a} = f'(c)$$

Traditional order of four big ideas with right emphasis:

1. Limits: Algebra of Inequalities
2. Derivatives: Ratios of Change
3. Integrals: Accumulation
4. Series: Sequences of Partial Sums

Preferred order of four big ideas with right emphasis:

1. Integrals: Accumulation
2. Derivatives: Ratios of Change
3. Series: Sequences of Partial Sums
4. Limits: Algebra of Inequalities

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